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One-Loop Amplitudes Of Gluons In SQCD

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One-loop amplitudes of gluons in supersymmetric Yang-Mills are four-dimensional cut-constructible. This means that they can be determined from their unitarity cuts. We present a new systematic procedure to explicitly carry out any finite unitarity cut integral. The procedure naturally separates the contributions from bubble, triangle and box scalar integrals. This technique allows the systematic calculation of $\mathcal{N} = 1$ amplitudes of gluons. As an application we compute all next-to-MHV six-gluon amplitudes in $\mathcal{N} = 1$ super-Yang-Mills.

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1. Introduction

Scattering amplitudes of gluons in Yang-Mills theories exhibit a remarkable simplicity that is not manifest from their calculation using Feynman diagrams. At tree-level, Parke-Taylor or maximally helicity violating (MHV) amplitudes [1] provide a striking example.

One-loop MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills also exhibit remarkable simplicity when computed using the unitarity based method [2,3,4]. Recently, a new technique was presented for computing general one-loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills using quadruple cuts [5]. This is a systematic and simple procedure that blends old [6,7], more modern [2,3,4,8,9,10] and very recent ideas [11,12] to uncover the simplicity of generic amplitudes. Using this one can easily reproduce all known results [2,3,13,14,10,15] and in principle compute any other amplitude.

One main motivation for computing $\mathcal{N} = 4$ amplitudes of gluons is that they are part of amplitudes in theories with less supersymmetry [2] (for a review, see [16]).

In this paper we concentrate on one-loop $\mathcal{N} = 1$ amplitudes of gluons. Such an amplitude can be decomposed as follows,

$$\mathcal{A}^{\mathcal{N}=1 \text{ vector}} = \mathcal{A}^{\mathcal{N}=4} - 3\mathcal{A}^{\mathcal{N}=1} \tag{1.1}$$

where $\mathcal{A}^{\mathcal{N}=4}$ is an amplitude where the full $\mathcal{N} = 4$ multiplet runs in the loop, and $\mathcal{A}^{\mathcal{N}=1}$ denotes the contribution from an $\mathcal{N} = 1$ chiral supermultiplet running in the loop. As mentioned above, the $\mathcal{N} = 4$ problem is easy to solve using quadruple cuts. On the other hand, $\mathcal{A}^{\mathcal{N}=1}$ only contains fermions and scalars in the loop and thus it is expected to be simpler than the full $\mathcal{N} = 1$ vector multiplet. This is why the decomposition (1.1) is useful.

The computation of $\mathcal{A}^{\mathcal{N}=1}$ is also important because it is part of a supersymmetry decomposition of a QCD amplitude at next-to-leading order,

$$\mathcal{A}^{\text{QCD}} = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1} + \mathcal{A}^{\text{scalar}} \tag{1.2}$$

where $\mathcal{A}^{\text{QCD}}$ denotes an amplitude with only a gluon running in the loop. $\mathcal{A}^{\text{scalar}}$ is an amplitude with only a complex scalar running in the loop.

The benefit of this approach is that supersymmetric amplitudes are four-dimensional cut-constructible [2,3]. This means that they can be completely determined by studying their finite unitarity cuts, and therefore the dimensional regularization parameter can be set to zero. Furthermore, they can be expressed as a linear combination of known integrals

called scalar box, triangle, and bubble integrals, with rational coefficients in the kinematical invariants.

The scalar part is more complicated, as only part of it can be determined by studying four-dimensional unitarity cuts. There are single-valued pieces that have to be determined using some other method.[1] The current state of the art in QCD is the five-gluon amplitude [18]. This means that even the scalar part has been fully computed for all helicity configurations.

For special helicity configurations much more is known. $\mathcal{A}^{\mathcal{N}=1}$ is known for all MHV amplitudes [3]. Also in [3], the cut constructible part of $\mathcal{A}^{\text{scalar}}$ was given for MHV amplitudes where the gluons of negative helicity are adjacent. More recently, the non-adjacent case was computed in [19] using the techniques of [11,20,21]. Also recently, $\mathcal{A}^{\mathcal{N}=1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ was presented in [22] and then extended to $\mathcal{A}^{\mathcal{N}=1}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ in [23]. Also in [23], the scalar box coefficients of all other next-to-MHV six-gluon amplitudes were computed using quadruple cuts [5].

For $\mathcal{A}^{\text{scalar}}$, apart from the five-gluon case, all amplitudes with at most one negative helicity gluon are also known [24,25,26].

In this paper, we introduce a systematic approach to computing any finite unitarity cut of amplitudes of gluons. Although our focus is on $\mathcal{A}^{\mathcal{N}=1}$, it is important to mention that this can also be applied to obtain the four-dimensional cut constructible part of $\mathcal{A}^{\text{scalar}}$ or even as an alternative way of computing $\mathcal{A}^{\mathcal{N}=4}$ amplitudes. The basic idea is to exploit the representation of the Lorentz invariant measure of a null vector ℓ, introduced in [20], as a measure over $\mathbb{R}^+ \times \mathbb{CP}^1 \times \mathbb{CP}^1$ with contour of integration a certain diagonal $\mathbb{CP}^1$. More explicitly, one writes $\ell_{a\dot{a}} = t\lambda_a \widetilde{\lambda}_{\dot{a}}$, and then

$$\int d^4\ell\, \delta^{(+)}(\ell^2)(\bullet) = \int_0^\infty t\, dt \int_{\widetilde{\lambda} = \overline{\lambda}} \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}](\bullet) \qquad (1.3)$$

where ($\bullet$) represents a generic integrand.

It turns out that the integration on the right hand side of (1.3) can always be reduced in a systematic way to an integral performed in [20]. The main simplification arises because the final integrals always localize to some poles in the region of integration. This is reminiscent of the technique developed in [13] where certain differential operators are applied to the cut integral in order to produce a localization via a holomorphic anomaly

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[1] In principle the whole scalar part can be computed from unitarity cuts if higher orders in ϵ, the dimensional regularization parameter, are kept. (See section 4.4 of [17].)

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[27]. Surprisingly, here we find that up to an integration over a single Feynman parameter, which is responsible for logarithms, all unitarity cuts localize by themselves without the need of a differential operator.

In the case of unitarity cuts of $\mathcal{N}=1$ amplitudes of gluons, $\mathcal{A}^{\mathcal{N}=1}$, one expects bubble, triangle and box scalar integrals to contribute to a given cut. Remarkably, our procedure naturally leads to a clean separation of the three kinds of contributions, allowing for an individual calculation of the corresponding coefficients.

An important simplification in $\mathcal{A}^{\mathcal{N}=1}$ is that one- and two-mass triangle coefficients never need to be computed. It turns out that their contributions always cancel against singular pieces in box integrals. This leaves us with bubbles, three-mass triangles and finite boxes, which we define in detail. Since the coefficient of scalar boxes can easily be computed from quadruple cuts [5] one can disregard that piece and concentrate on the bubble and three-mass triangle scalar integral coefficients.

As an application of our technique we compute all $\mathcal{A}^{\mathcal{N}=1}$ next-to-MHV six-gluon amplitudes. The reason we have undertaken the whole calculation is because six-gluon amplitudes in QCD are going to be important for future colliders and our computation completes the second piece in (1.2). It is important to mention that this calculation requires the use of tree-level amplitudes of gluons with two fermions or two scalars. Luckily, very compact formulas for those amplitudes were derived very recently [28,29] by extending the techniques of [30,31].

This paper is organized as follows: In section 2, we define $\mathcal{A}^{\mathcal{N}=1}$ in terms of a linear combination of scalar integrals. Using its singular behavior we show that only bubble, three-mass triangle and finite boxes are necessary. In section 3, we study general unitarity cut integrals and present the method for computing them explicitly. For $\mathcal{A}^{N=1}$ we explain the way to basically read off the coefficients of bubbles and three-mass triangles. In section 4, we present the calculation of $\mathcal{A}^{\mathcal{N}=1}$ non-MHV six-gluon amplitudes. We also present our results for the next-to-MHV n-gluon amplitude where all negative-helicity gluons are consecutive. This amplitude has appeared in [23], but by our procedure it emerges in a different form. Explicit details are given in order to illustrate the steps described in section 3. In section 5, we summarize the results for all the amplitudes computed in section 4. Section 5 is intended to be self contained so that the reader interested only in the six-gluon amplitude results can skip the rest of the paper. Appendix A contains a detailed definition of scalar integrals as well as our definition of finite box integrals. Appendix B summarizes the results of tree-level amplitudes of gluons and fermions needed in section 4.

Throughout the paper, we use the following notation and conventions along with those of [11] and the spinor helicity-formalism [32,33,34]. The external gluon labeled by i carries momentum p_i. Since $p_i^2 = 0$, it can be written as a bispinor $(p_i)_{a\dot{a}} = \lambda_{i\ a}\widetilde{\lambda}_{i\ \dot{a}}$. Inner product of null vectors $p_{a\dot{a}} = \lambda_a\widetilde{\lambda}_{\dot{a}}$ and $q_{a\dot{a}} = \lambda'_a\widetilde{\lambda}'_{\dot{a}}$ can be written as $2p \cdot q = \langle \lambda, \lambda' \rangle [\widetilde{\lambda}, \widetilde{\lambda}']$, where $\langle \lambda, \lambda' \rangle = \epsilon_{ab}\lambda^a \lambda'^b$ and $[\widetilde{\lambda}, \widetilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}^{\dot{a}}\widetilde{\lambda}'^{\dot{b}}$. Other useful definitions are:

$$P_{i\ldots j} \equiv p_i + p_{i+1} + \cdots + p_j$$
$$K_i^{[r]} \equiv p_i + p_{i+1} + \cdots + p_{i+r-1}$$
$$t_i^{[r]} \equiv (p_i + p_{i+1} + \cdots + p_{i+r-1})^2$$
$$\langle i| \sum_r p_r |j] \equiv \sum_r \langle i\ r \rangle [r\ j]$$
$$\langle i| (\sum_r p_r)(\sum_s p_s)|j\rangle \equiv \sum_r \sum_s \langle i\ r \rangle [r\ s]\langle s\ j \rangle$$
$$[i|(\sum_r p_r)(\sum_s p_s)|j] \equiv \sum_r \sum_s [i\ r]\langle r\ s \rangle [s\ j]$$
$$\langle i|(\sum_r p_r)(\sum_s p_s)(\sum_t p_t)|j] \equiv \sum_r \sum_s \sum_t \langle i\ r \rangle [r\ s]\langle s\ t \rangle [t\ j]$$

(1.4)

where addition of indices is always done modulo n.

2. One-Loop $\mathcal{N} = 1$ Amplitudes

Amplitudes of gluons at one-loop admit a color decomposition [35,36] with single and double trace contributions. The piece proportional to the single trace term Tr $(T^{a_1} \ldots T^{a_n})$ is called the leading color partial amplitude and it is denoted by $A_{n;1}(1, \ldots, n)$. In this paper we concentrate on $A_{n;1}(1, \ldots, n)$. The reason is that when all particles in the loop are in the adjoint representation, all sub-leading color amplitudes are given as linear combinations of $A_{n;1}$ with permutations of the gluon labels (See section 7 of [2] for a proof.) This is the case for all amplitudes we consider. In the remainder of the paper we will simplify the subscript and just denote the leading color partial amplitude by $A_n(1, \ldots, n)$.

We consider amplitudes of gluons where an $\mathcal{N} = 1$ chiral multiplet circulates in the loop. Reduction techniques allow us to express these amplitudes in terms of scalar integrals in the shapes of boxes I_4, triangles I_3, and bubbles I_2 [37,3]. These functions are given explicitly in appendix A, along with some helpful figures.

The amplitude thus takes the following form:

$$A_n^{\mathcal{N}=1} = \frac{r_\Gamma(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \sum \left(c_4^{1m} I_4^{1m} + c_4^{2m\ e} I_4^{2m\ e} + c_4^{2m\ h} I_4^{2m\ h} + c_4^{3m} I_4^{3m} + c_4^{4m} I_4^{4m} \right.$$
$$\left. + c_3^{1m} I_3^{1m} + c_3^{2m} I_3^{2m} + c_3^{3m} I_3^{3m} + c_2 I_2 \right). \tag{2.1}$$

Here $\epsilon = (4 - D)/2$ is the dimensional regularization parameter, μ is the renormalization scale, and r_Γ is defined by

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon).} \tag{2.2}$$

The sum runs over all the cyclic permutations within each type of integral. The coefficients c of the scalar integrals are rational functions of spinor products. This follows from the reduction procedure [3].

2.1. Singular Behavior

The infrared and ultraviolet singular behavior of these amplitudes is known and was given in [38,39,40]. For the case of a gluon amplitude with the $\mathcal{N} = 1$ chiral multiplet in the adjoint representation circulating in the loop, the divergent behavior is given simply in terms of the tree-level amplitude by

$$A_n^{\mathcal{N}=1}|_{\text{singular}} = \frac{r_\Gamma}{\epsilon(4\pi)^{2-\epsilon}} A_n^{\text{tree}}, \tag{2.3}$$

where

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{2.4}$$

and A_n^{tree} is the color-ordered tree-level amplitude.

Now let us see what this means for the scalar integral coefficients. The integrals I_3^{3m} and I_4^{4m} are finite. Therefore their coefficients do not contribute to (2.3). The bubble integral diverges as $1/\epsilon$. One- and two-mass triangle integrals can be conveniently written in terms of the function

$$T(s) = \frac{r_\Gamma}{\epsilon^2}(-s)^{-\epsilon} \tag{2.5}$$

as follows:

$$I_3^{1m}(s) = \frac{1}{(-s)} T(s), \qquad I_3^{2m}(s,t) = \frac{1}{(-s)-(-t)} \left(T(s) - T(t) \right) \tag{2.6}$$

where s and t denote the invariants in different independent channels.

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Finally, one-, two-, and three-mass box scalar integrals have the property that they can be made finite by adding linear combinations of $T(s)$ functions. We denote the finite box integral functions by I_{4F}, where F stands for finite. Their definition is given in detail in appendix A.

Now we are ready to derive the main result of this section. From (2.3) we see that all divergences of the form $1/\epsilon^2$ must be absent. This implies that all $T(s)$ functions must cancel among the different terms. Since one- and two-mass triangle integrals are given entirely as linear combination of $T(s)$ functions with rational coefficients, it follows that their coefficients are such that they do not appear in the final answer for the amplitude. Our interpretation is that the only reason they must be included is to cancel the $1/\epsilon^2$ divergences from the box integral.

Therefore, we reach the conclusion that $\mathcal{A}_n^{\mathcal{N}=1}$ can be written as a linear combination of finite box scalar integrals I_{4F}, three-mass triangles I_3^{3m} and bubbles I_2. More explicitly,

$$\mathcal{A}_n^{\mathcal{N}=1} = \frac{r_\Gamma (\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \sum \left(c_2 I_2 + c_3^{3m} I_3^{3m} + c_4^{1m} I_{4F}^{1m} + c_4^{2m\ e} I_{4F}^{2m\ e} + c_4^{2m\ h} I_{4F}^{2m\ h} + c_4^{3m} I_{4F}^{3m} + c_4^{4m} I_4^{4m} \right).$$

(2.7)

Finally, among the bubble coefficients there is one relation that must hold in order to satisfy (2.3), namely that the sum of all bubble coefficients reproduces the tree-level amplitude:

$$\sum c_2 = A_n^{\text{tree}}.$$

(2.8)

In section 4, (2.8) is used as a very non-trivial consistency check of our results for next-to-MHV six gluon amplitudes.

3. Coefficients from Unitarity Cuts

In this section we introduce a new method for computing explicitly any finite unitarity cut in a gauge theory with massless particles running in the loop. Of course, our aim here is to apply the technique to the computation of the coefficients in (2.7) which determine $\mathcal{A}^{\mathcal{N}=1}$. Nevertheless, it is important to mention that this can also be applied to obtain the four-dimensional cut constructible part of $\mathcal{A}^{\text{scalar}}$ or even as an alternative way of computing $\mathcal{A}^{\mathcal{N}=4}$ amplitudes.

The unitarity cut in the $(i, i+1, \ldots, j-1, j)$-channel is computed by cutting two propagators in the loop whose momenta differ by $P_{ij} = p_i + \ldots + p_j$ in all Feynman diagrams contributing to the amplitude. Adding up all these contribution we find a "cut integral" [41][2]

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[2] Further information about this technique may be found in [7]. This body of work was not
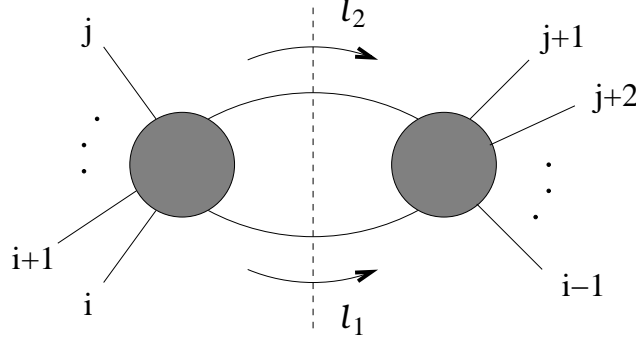
Fig. 1: Representation of the cut integral. Left and right tree-level amplitudes are on-shell. Internal lines represent the legs coming from the cut propagators.

$$C_{i,i+1,\ldots,j-1,j} =$$
$$\int d\mu A^{\text{tree}}(\ell_1, i, i+1, \ldots, j-1, j, \ell_2) A^{\text{tree}}((-\ell_2), j+1, j+2, \ldots, i-2, i-1, (-\ell_1)),$$
$$(3.1)$$

where $d\mu = d^4\ell_1 d^4\ell_2 \delta^{(+)}(\ell_1^2)\delta^{(+)}(\ell_2^2)\delta(\ell_1 + \ell_2 - P_{ij})$ is the Lorentz invariant phase space measure of two light-like vectors (ℓ_1, ℓ_2) constrained by momentum conservation. See fig. 1.

This cut integral computes the discontinuity of the amplitude across a given branch cut in the space of kinematical invariants. The same discontinuity can be computed from the right hand side of (2.1). Note that the coefficients are rational and thus do not have branch cuts. The scalar integrals have a cut in this channel only if they contain the same two propagators that are cut.

The idea is to determine the cuts of the known scalar integrals, compute (3.1) explicitly and then solve for the coefficients by comparing both sides.

The phase space integral in (3.1) can be performed following the techniques of [20]. The idea is to use momentum conservation to write the integral entirely in terms of just one of the cut propagator momenta, say $\ell = \ell_1$. This vector is then parametrized as $\ell_{a\dot{a}} = t\lambda_a \widetilde{\lambda}_{\dot{a}}$, where the scale t is real and the spinors λ and $\widetilde{\lambda}$ are independent homogeneous coordinates on two copies of $\mathbb{CP}^1$. The integral is then performed over the diagonal $\mathbb{CP}^1$ defined by $\widetilde{\lambda} = \overline{\lambda}$. The integral can be rewritten as

$$\int d^4\ell \delta^{(+)}(\ell^2)\,(\bullet) = \int_0^\infty dt\, t \int \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}](\bullet), \qquad (3.2)$$

intended to apply to massless theories. We find that the material can nevertheless be adapted for the considerations of this paper. The modern interpretation is found in [2,3].

where the bullets represent generic arguments.

We first illustrate the procedure by computing the cuts of bubble and three-mass triangle scalar integrals entering in (2.7) and then we discuss the calculation of the general cut integral (3.1).

3.1. Bubble and Three-Mass Triangle Unitarity Cuts

As a warm-up let us compute the double cut of a bubble and a three-mass triangle using (3.2). We will need these results to read off the coefficients we need. Let us denote the cut of a scalar integral I by ΔI.

The unitarity cut of a bubble is given by (see appendix A for a definition of the original integral[3])

$$
\begin{aligned}
\Delta I_2(K) &= \int d^4\ell\, \delta^{(+)}(\ell^2)\delta^{(+)}((\ell-K)^2) \\
&= \int_0^\infty t\, dt \int \langle \lambda\, d\lambda\rangle [\widetilde{\lambda}\, d\widetilde{\lambda}]\delta^{(+)}(K^2 - tK_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}}) \\
&= \int \langle \lambda\, d\lambda\rangle [\widetilde{\lambda}\, d\widetilde{\lambda}]\frac{K^2}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2}
\end{aligned}
\tag{3.3}
$$

were we have used that in the kinematic regime where $K^2 > 0$ the delta function always has its support in the integration region of t.

We postpone evaluating the last integral in (3.3). First let us evaluate a slightly more complicated integral [20][4],

$$
\mathcal{I} = \int \langle \lambda\, d\lambda\rangle [\widetilde{\lambda}\, d\widetilde{\lambda}]\frac{1}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2}g(\lambda)
\tag{3.4}
$$

with

$$
g(\lambda) = \frac{\prod_{i=1}^k \langle \lambda, A_i\rangle}{\prod_{j=1}^k \langle \lambda, B_j\rangle}
\tag{3.5}
$$

First note the following identity that holds for an arbitrary but fixed negative chirality spinor η:

$$
\frac{[\widetilde{\lambda}\, d\widetilde{\lambda}]}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2}g(\lambda) = -d\widetilde{\lambda}^{\dot{c}}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{c}}}\left(\frac{[\widetilde{\lambda},\eta]}{(K_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})(K_{a\dot{a}}\lambda^a\eta^{\dot{a}})}g(\lambda)\right).
\tag{3.6}
$$

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[3] In the following calculations, we are omitting the factor of $-i\frac{(4\pi)^{2-\epsilon}}{(2\pi)^{4-2\epsilon}}$ which has been accounted for in the overall factor in (2.1) and (2.7).

[4] This integral was performed as part of a derivation of MHV diagrams for tree-level amplitudes of gluons from a twistor string theory calculation [11].

This identity holds for all values of λ except for those where the denominator vanishes along the contour of integration. The reason is that along the contour of integration $\widetilde{\lambda} = \overline{\lambda}$ and therefore

$$-d\widetilde{\lambda}^{\dot{c}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{c}}} \frac{1}{\langle \lambda, \zeta \rangle} = 2\pi \overline{\delta}(\langle \lambda, \zeta \rangle), \tag{3.7}$$

where we have introduced a $(0,1)$-form $\overline{\delta}(\langle \lambda, \zeta \rangle)$, such that

$$\int \langle \lambda \, d\lambda \rangle \, \overline{\delta}(\langle \lambda, \zeta \rangle) B(\lambda) = -iB(\zeta). \tag{3.8}$$

Let us write the complete form of (3.6) that is valid for all values of λ along the contour of integration:

$$\frac{[\widetilde{\lambda} \, d\widetilde{\lambda}]}{(K_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}})^2} g(\lambda) = - \, d\widetilde{\lambda}^{\dot{c}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{c}}} \left(\frac{[\widetilde{\lambda}, \eta]}{(K_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}})(K_{a\dot{a}} \lambda^a \eta^{\dot{a}})} g(\lambda) \right)$$
$$+ \frac{2\pi [\widetilde{\lambda}, \eta]}{K_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}}} \left(-\overline{\delta}(K_{a\dot{a}} \lambda^a \eta^{\dot{a}}) g(\lambda) + \frac{1}{K_{a\dot{a}} \lambda^a \eta^{\dot{a}}} \sum_{j=1}^{k} \overline{\delta}(\langle \lambda, B_j \rangle) g(\lambda) \langle \lambda, B_j \rangle \right). \tag{3.9}$$

The contribution from the first term in (3.9) gives zero after integration over λ. On the other hand, the delta functions in the remaining terms localize the λ integral and give the value of the integrand at the pole. More explicitly one uses that

$$\int \langle \lambda, d\lambda \rangle \overline{\delta}(\langle \lambda, \lambda_B \rangle) H(\lambda) = -iH(\lambda_B). \tag{3.10}$$

A short calculation reveals that (3.4) is given by

$$\mathcal{I} = -\frac{1}{K^2} g(\lambda_K) + \sum_{j=1}^{k} \frac{[B_j, \eta]}{\langle B_j | K | B_j] \langle B_j | K | \eta]} \frac{\prod_{i=1}^{k} \langle B_j, A_i \rangle}{\prod_{l \neq j} \langle B_j, B_l \rangle} \tag{3.11}$$

where $\lambda_{K\,a} = K_{a\dot{a}} \eta^{\dot{a}}$.

This is the basic result that will allow us to calculate any double cut in section 3.2.

Going back to the cut of the bubble integral (3.3) we find that by setting $g(\lambda) = 1$ in (3.11)

$$\Delta I_2(K) = -1. \tag{3.12}$$
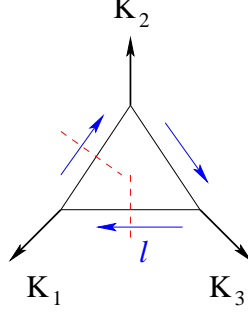
Consider now a three-mass triangle integral.

Fig. 2: A double cut of a three-mass triangle integral.

Denote the momenta at the vertices by K_1, K_2 and K_3. Let us calculate the cut in the K_1 channel, Δ_1. Let the momentum in the cut propagators be ℓ and $\ell - K_1$. See fig. 2. Then

$$
\begin{aligned}
\Delta_1 I_3^{3m} &= -\int d^4\ell \, \delta^{(+)}(\ell^2) \frac{\delta^{(+)}((\ell - K_1)^2)}{(\ell + K_3)^2} \\
&= -\int_0^\infty t \, dt \int \langle \lambda \, d\lambda \rangle [\widetilde\lambda \, d\widetilde\lambda] \frac{\delta(K_1^2 - t K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a})}{K_3^2 + t K_{3,a\dot a} \lambda^a \widetilde\lambda^{\dot a}} \\
&= -\int \langle \lambda \, d\lambda \rangle [\widetilde\lambda \, d\widetilde\lambda] \frac{K_1^2}{(K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a})^2} \frac{(K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a})}{K_3^2 (K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a}) + K_1^2 (K_{3,a\dot a} \lambda^a \widetilde\lambda^{\dot a})} \\
&= -\int \langle \lambda \, d\lambda \rangle [\widetilde\lambda \, d\widetilde\lambda] \frac{1}{(K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a})(Q_{a\dot a} \lambda^a \widetilde\lambda^{\dot a})}
\end{aligned}
\tag{3.13}
$$

where $Q_{a\dot a} = \frac{K_3^2}{K_1^2}(K_{1,a\dot a}) + (K_{3,a\dot a})$.

This integral can be brought to the same form as that in (3.3) by introducing a Feynman parameter to combine the two denominators into one. This Feynman parameter integration turns out to be the only one needed in the calculation of general cut integrals in section 3.2. This is why we show it explicitly here.

$$
\frac{1}{(K_{1,a\dot a} \lambda^a \widetilde\lambda^{\dot a})(Q_{a\dot a} \lambda^a \widetilde\lambda^{\dot a})} = \int_0^1 dx \frac{1}{(((1-x)K + xQ)_{a\dot a} \lambda^a \widetilde\lambda^{\dot a})^2}
\tag{3.14}
$$

Performing first the λ and $\widetilde\lambda$ integrations, we get

$$
\int_0^1 dx \frac{1}{((1-x)K + xQ)^2}.
\tag{3.15}
$$

The result of the x integration in the kinematic regime corresponding to the channel under consideration, i.e., where $K_1^2 > 0$ and both $K_2^2 < 0, K_3^2 < 0$ is then:

$$
\Delta_1 I_3^{3m} = \frac{1}{\sqrt\Delta} \left(\ln \frac{2Q^2 - \sqrt\Delta}{2Q^2 + \sqrt\Delta} - \ln \frac{2(-K_2 \cdot K_3 - K_1^2) - \sqrt\Delta}{2(-K_2 \cdot K_3 - K_1^2) + \sqrt\Delta} \right),
\tag{3.16}
$$

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where Δ denotes the discriminant that arises from the quadratic equation in the Feynman parameter, given by

$$\Delta = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2K_1^2 K_2^2 - 2K_2^2 K_3^2 - 2K_3^2 K_1^2. \qquad (3.17)$$

3.2. General Cut Integrals

As discussed in section 2, calculating all one-loop $\mathcal{N} = 1$ amplitudes is equivalent to finding the rational coefficients of boxes, three-mass triangles and bubbles. The box coefficients can be computed by quadruple cuts. In principle, the three-mass triangle coefficients can be computed using triple cuts and the bubble coefficients require double cuts. In practice we find that all bubble and three-mass triangle coefficients can be computed from double cuts in a simple and systematic manner. It turns out that the separation of the three-mass and bubble integral coefficients is easily done because of the striking difference in the form of their cuts; see (3.12) and (3.16).

In the remainder of this section, we explain how to perform general double cut integrals (3.1).

First, find spinor-product expressions for the two tree-level amplitudes that form the integrand of (3.1). These are tree-level amplitudes of gluons with two fermions or two scalars. Two recent techniques allow the calculation of those amplitudes: MHV diagrams [42,43,44,45] and recursion relations [28,29]. However, at this point all we need is that they are rational functions in the spinor products.

Recall that the measure in (3.1) is $d\mu = d^4\ell_1 d^4\ell_2 \delta^{(+)}(\ell_1)\delta^{(+)}(\ell_2)\delta^{(4)}(\ell_1 + \ell_2 - P_{ij})$, where $P_{ij} = p_i + p_{i+1} + \ldots + p_j$. Using the last delta function to perform the ℓ_2 integration and (3.2) to write the measure over ℓ_1, we find

$$C_{i,\ldots,j} = \int_0^\infty t\,dt \langle \lambda, d\lambda\rangle [\widetilde{\lambda}, d\widetilde{\lambda}] \delta^{(+)}(t\lambda_a \widetilde{\lambda}_{\dot{a}} P_{ij}^{a\dot{a}} - P_{ij}^2) G(\lambda, \widetilde{\lambda}, t). \qquad (3.18)$$

We denote ℓ_1 by ℓ when there is no possibility of confusion. Recall that $\ell_{a\dot{a}} = t\lambda_a \widetilde{\lambda}_{\dot{a}}$. $G(\lambda, \widetilde{\lambda}, t)$ is the function that arises from the product of the two tree-level amplitudes in (3.1). A simple observation that helps in actual calculations is that in order to obtain $G(\lambda, \widetilde{\lambda}, t)$, one has to write expressions of the form $\langle \bullet, \ell_2 \rangle$ or $[\bullet, \ell_2]$ in terms of $\ell = \ell_1$. A systematic way of doing this is by using the following identity:

$$\langle \bullet, \ell_2 \rangle = \frac{\langle \bullet, \ell_2 \rangle [\ell_2, \ell_1]}{[\ell_2, \ell_1]} = \frac{\langle \bullet |\ell_2| \ell_1]}{[\ell_2, \ell_1]} = \frac{\langle \bullet |P_{ij}| \ell_1]}{[\ell_2, \ell_1]}. \qquad (3.19)$$

A similar identity is valid for $[\bullet, \ell_2]$. The factors $[\ell_2, \ell_1]$ and $\langle \ell_2, \ell_1 \rangle$ all pair up in the end allowing for the use of the vector form of ℓ_2. This happens because the product of the amplitudes must be invariant under the scaling $z\lambda_{\ell_2}$ and $z^{-1}\widetilde{\lambda}_{\ell_2}$.

Going back to the integral (3.18), let us perform the t integration by using the delta function,

$$C_{i,\ldots,j} = P_{ij}^2 \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{(P_{ij}^{a\dot{a}} \lambda_a \widetilde{\lambda}_a)^2} G\left(\lambda, \widetilde{\lambda}, \frac{P_{ij}^2}{P_{ij}^{a\dot{a}} \lambda_a \widetilde{\lambda}_a}\right). \tag{3.20}$$

Let us assume that the $\widetilde{\lambda}$ dependence in the denominator of G is simpler, i.e. it has fewer factors than that of λ. If the opposite were true, we would use the conjugate of the discussion that follows.

Since λ and $\widetilde{\lambda}$ are independent homogeneous coordinates on two $\mathbb{CP}^1$ one must require t to transform as $t \to (wz)^{-1} t$ when $(\lambda, \widetilde{\lambda}) \to (w\lambda, z\widetilde{\lambda})$ so that $\ell_{a\dot{a}} = t\lambda_a \widetilde{\lambda}_{\dot{a}}$ remains invariant. For the integral (3.20) to make sense, it must be the case that $G(\lambda, \widetilde{\lambda}, t)$ is invariant under the scaling $(\widetilde{\lambda}, t) \to (z\widetilde{\lambda}, z^{-1}t)$. This ensures that G in (3.20) has degree zero in $\widetilde{\lambda}$. This implies that it can be written as a sum of terms of the form

$$\frac{\prod_l [A_l, \ell]}{\prod_i \langle \ell | Q_i | \ell] \prod_j [A_j, \ell]} g(\lambda). \tag{3.21}$$

Each term has degree zero in $\widetilde{\lambda}$. The function $g(\lambda)$ contains all other terms that do not depend on $\widetilde{\lambda}$.

There are two ways to proceed at this point. One is based on the introduction of several Feynman integration parameters. We find that this method becomes very cumbersome as the number of Feynman parameters increases. The second approach keeps the number of Feynman integrations to be at most one, and it has the advantage of leading to a clean separation of bubble, three-mass triangle and box coefficients. We discuss both approaches because they might be useful in different situations.

Feynman Parametrizations

We want to transform the denominator of (3.21) by replacing every factor $1/[A_i, \ell]$ by $-\langle A_i, \ell \rangle / \langle \ell | A_i | \ell]$ and then using Feynman parameters to combine all factors in the denominator, including the one in (3.20), into one of the form $1/\langle \ell | T(x_1, \ldots, x_{m+2}) | \ell]^{m+2}$.

The integral to be performed has now the form

$$\int \prod_{i=1}^{m+2} dx_i \delta\left(\sum_{j=1}^{m+2} x_j - 1\right) \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{\langle \ell | T(x_1, \ldots, x_{m+2}) | \ell]^{m+2}} \prod_{l=1}^{m} [A_l, \ell] \widetilde{g}(\lambda) \tag{3.22}$$

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where we have absorbed all $\langle A_i, \ell \rangle$ into $\widetilde{g}(\lambda)$. Recall that the total degree in $\widetilde{\lambda}$ of the integrand must be -2. Therefore, there are m factors in the numerator containing $\widetilde{\lambda}$.

It is not difficult to find an analog of (3.6), i.e., to write the integrand as a total derivative in $\widetilde{\lambda}$. Indeed, one can prove by induction that

$$
\frac{[\widetilde{\lambda} \, d\widetilde{\lambda}] \prod_{i=1}^{j} [A_i \, \ell][\eta \, \ell]^{m-j}}{\langle \ell | T | \ell \rangle^{m+2}}
$$
$$
= [d\ell \, \partial_\ell] \left[\frac{\prod_{i=1}^{j} \langle \ell | T | A_i \rangle}{\langle \ell | T | \ell \rangle^{m+1}} \left(\sum_{k=0}^{j} \frac{(-1)^{j-k} (j-k)!}{(m+1-j)...(m+1-k)} g_k[x_i] \frac{[\eta \, \ell]^{m+1-k}}{\langle \ell | T | \eta \rangle^{j+1-k}} \right) \right].
\tag{3.23}
$$

Here, η is an arbitrary but fixed spinor and

$$
g_k[x_s] = \sum_{i_1 < i_2 < ... < i_k} x_{i_1} \ldots x_{i_k} \quad \text{with} \quad x_i = \frac{[A_i, \ell]}{\langle \ell | T | A_i \rangle}.
\tag{3.24}
$$

Now we can proceed as explained in section 3.1. The idea is to realize that (3.23) is valid for all values of λ except those for which there is a pole. The final value of the integral comes entirely from those poles as in (3.9) that led to (3.11).

The complication with this approach is that the final result is expressed in terms of several Feynman parameter integrations. In practice, if the number of Feynman parameters is two or more then the integration procedure is cumbersome.

Simple Pole Expansion

In order to avoid the proliferation of Feynman parameters we propose a second way of treating the integral (3.20). The idea is that before doing the integral, we should separate the denominator factors with $\widetilde{\lambda}$ as much as possible, at the cost of more terms. Where there is a product $[a \, \ell][b \, \ell]$ in the denominator, multiply both numerator and denominator by $[a \, b]$. Then apply Schouten's identity,

$$
[i \, j][k \, l] = [i \, k][j \, l] + [i \, l][k \, j],
\tag{3.25}
$$

in the numerator with another factor $[c \, \ell]$ (which must exist by homogeneity). We then get two terms with $[a \, \ell]$ or $[b \, \ell]$ in the numerator, cancelling one of the denominator factors. The result, in terms of $\widetilde{\lambda}_\ell$, is a denominator of the form $\prod_r \langle \ell | Q_r | \ell \rangle [A \, \ell]$ in every term.

Factors of the form $\langle \ell | Q_r | \ell \rangle$ can be treated in the same way by writing

$$
\langle \ell | Q_r | \ell \rangle = [\widetilde{Q}_r, \ell]
\tag{3.26}
$$

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where $\widetilde{\lambda}_{Q_r \, \dot{a}} = -(Q_r)_{a\dot{a}} \lambda_\ell^a$.

Using this procedure to split poles in $\widetilde{\lambda}$ and given that the integrand has degree -2 in $\widetilde{\lambda}$, one finds in the end only two possible kind of integrals

$$\mathcal{I}_A = \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{(P_{ij}^{a\dot{a}} \lambda_a \widetilde{\lambda}_a)^2} H(\lambda); \qquad \mathcal{I}_B = \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{(P_{ij}^{a\dot{a}} \lambda_a \widetilde{\lambda}_a)^2} \frac{[B, \ell]}{\langle \ell | Q_r | \ell]} H(\lambda). \qquad (3.27)$$

For the second class of integrals we can apply the splitting procedure once more to split the product $\langle \ell | P | \ell] \langle \ell | Q_r | \ell]$. Then we find an integral of the form $\mathcal{I}_A$ and one of the form

$$\mathcal{I}_C = \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{\langle \ell | P | \ell] \langle \ell | Q_r | \ell]} H(\lambda). \qquad (3.28)$$

Here $H(\lambda)$ is used to denote a generic function of ℓ and independent of $\widetilde{\lambda}$.

As anticipated in section 3.1, the two kind of integrals, i.e., $\mathcal{I}_A$ and $\mathcal{I}_C$, appeared in the calculation of the bubble and three-mass triangle integral discontinuities. Therefore one can repeat the same procedure for their computation. In particular, $\mathcal{I}_A$ does not require any Feynman parameters and produces a rational function. On the other hand, $\mathcal{I}_C$ only requires one Feynman parameter and produces only logarithms.

Canonical Decomposition

The decomposition of the cut integral into integrals of the form $\mathcal{I}_A$ and $\mathcal{I}_C$ has a very useful byproduct. Since $\mathcal{I}_A$ produces only rational functions and $\mathcal{I}_C$ produces only logarithms, it is easy to conclude that the bubble coefficient is given by

$$c_2 = \int \frac{\langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]}{(P_{ij}^{a\dot{a}} \lambda_a \widetilde{\lambda}_a)^2} H(\lambda). \qquad (3.29)$$

Here we have used the fact that for a given unitarity cut, there is only one bubble integral with the corresponding branch cut.

In the calculation of $\mathcal{I}_C$ one has to introduce a Feynman parameter x to write the factors with $\widetilde{\lambda}$ in the denominator as $\langle \ell | xP + (1-x)Q | \ell]^2$. The integration over λ and $\widetilde{\lambda}$ produces two different kind of terms. The first comes from the pole $\langle \ell | xP + (1-x)Q | \eta]$. The second kind comes from the poles in $H(\lambda)$. This is explained in detail in section 3.1 where (3.11) is computed. There the two kind of terms are shown explicitly.

Let us write the contribution to $\mathcal{I}_C$ from the first term in (3.11),

$$\int_0^1 dx \frac{1}{(xP + (1-x)Q)^2} H(\lambda(x)) \qquad (3.30)$$

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where $\lambda(x)$ is the solution of $\langle\ell|xP + (1-x)Q|\eta] = 0$.

It is easy to see that the contributions to $\mathcal{I}_C$ from the poles of $H(\lambda)$ will only have linear factors in x in the denominator. This implies that upon integration in x they can only produce logarithms of rational functions of the kinematical invariants. These simple functions come from the discontinuity of one-, two-, and three-mass scalar box integrals.

On the other hand, the term given in (3.30) can produce a more complicated object. There are again two cases: if the discriminant of the quadratic equation $(xP + (1-x)Q)^2 = 0$ is a perfect square, then we find more one-, two-, and three-mass scalar box contributions. If the discriminant is not a perfect square then we either have the contribution of a three-mass triangle or a four-mass box integral.

Recall that the coefficient of scalar box integrals can be computed very efficiently by using the quadruple cut technique introduced in [5]. This implies that we can ignore all those contributions and concentrate only on the three-mass triangle coefficients.

The discriminant of $(xP + (1-x)Q)^2$ is given by

$$\Delta = 4(((P-Q) \cdot Q)^2 - (P-Q)^2 Q^2). \tag{3.31}$$

Therefore, we are only interested in integrals that produce a discriminant of the form (3.17), i.e.

$$\Delta_{3m} = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2K_1^2 K_2^2 - 2K_3^2 K_1^2 - 2K_2^2 K_3^2. \tag{3.32}$$

From the calculation of the three-mass triangle cut in section 3.1 we know that if $P = K_1$ then in order to produce (3.32) we need $Q_{a\dot{a}} = \frac{K_3^2}{K_1^2}(K_1)_{a\dot{a}} + (K_3)_{a\dot{a}}$.

Once we have identified the integral that has the discriminant of a three-mass triangle scalar integral we have to perform one more decomposition. The idea is to expand the integrand in simple fractions, as a function of x, until we find a term of the form

$$c_3^{3m} \int_0^1 dx \frac{1}{(xP + (1-x)Q)^2} \tag{3.33}$$

whose coefficient we can identify with the three-mass triangle coefficient c_3^{3m}. This procedure is applied in detail in the calculation of the cut $C_{23}^{(3;3)}$ in section 4.2.

We illustrate all the features of this procedure for calculating bubble and three-mass triangle coefficients in the next section with the example of the six-gluon amplitude. One particularly challenging technical point is that generically one might expect poles in the denominator of the form $\langle\ell|PQ|\ell\rangle$. This is a quadratic equation for λ. We explain how to deal with this in the calculation of the coefficient $c_{2;2;2}^{(3)}$ in section 4.2.

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4. Example: All $\mathcal{N}=1$ Next-To-MHV Six-Gluon Amplitudes

There are several pieces missing in the calculation of the six-gluon next-to-leading order scattering amplitude in QCD. They are: next-to-MHV $A^{\mathcal{N}=1}$, MHV and next-to-MHV A^{scalar}. Recently important progress has been made for the first class of amplitudes. In [22], the amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ was presented. In [46], all scalar box coefficients of the remaining helicity configurations were computed using quadruple cuts. It is the aim of this section to present all the remaining coefficients and some new forms for known ones. These results complete the next-to-MHV $A^{\mathcal{N}=1}$ piece.

4.1. First Configuration: $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

This amplitude has been computed in [22]. Here we rederive it to illustrate our integration technique. Our result will agree with [22] but emerge in a slightly different form.

The first observation is that all the box and triangle coefficients vanish. This is easily seen by examining the helicity assignments in all possible distributions.

Thus every double cut simply gives the coefficient of the associated bubble integral. The nonvanishing cuts are C_{34}, C_{61}, C_{234} and C_{345}. Among these, cuts C_{234} and C_{345} are mapped to each other by the permutation of indices $P_\alpha : 1 \leftrightarrow 6, 2 \leftrightarrow 5, 3 \leftrightarrow 4$ plus conjugation, while cuts C_{34} and C_{61} are invariant under P_α. There is another permutation $P_\beta : 1 \leftrightarrow 3, 4 \leftrightarrow 6$ under which C_{234} and C_{34} are mapped to C_{345} and C_{61} respectively. So there are only two independent integrands which are given as

$$
\begin{aligned}
C_{34} &= \int d\mu A^{\text{tree}}(\ell_1, 5, 6, 1, 2, \ell_2) A^{\text{tree}}((-\ell_2), 3, 4, (-\ell_1)) \\
&= \int d\mu \frac{\langle 3|1+2|6]^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 5|6+1|2] P_{612}^2} \frac{\langle \ell_1|1+2|6][\ell_2\ 4]}{\langle \ell_1\ 5\rangle[\ell_2\ 3]} \\
&\quad + \int d\mu \frac{\langle 1|5+6|4]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[3\ 4]\langle 5|6+1|2] P_{561}^2} \frac{\langle 1|5+6|\ell_2]\langle 3\ \ell_1\rangle}{[2\ \ell_2]\langle 4\ \ell_1\rangle}
\end{aligned}
\tag{4.1}
$$

and

$$
C_{612} = -\int d\mu \frac{\langle 3|P_{345}|6]^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P_{345}^2} \frac{\langle \ell_1|P_{345}|6]\langle 3\ \ell_1\rangle}{\langle \ell_1|P_{345}|2]\langle 5\ \ell_1\rangle}
\tag{4.2}
$$

C_{612}. When we calculate the integrand, we need to use the tree-level amplitude of four gluons and a pair of fermions and complex scalars. These amplitudes are given in Appendix B.

Now we will do the integration. To demonstrate our method, we will do one integration (for the cut C_{34}) in detail and then simply cite the other three results, which are obtained similarly.

The Cut C_{34}:

The integrand is given by (4.1). There are two terms which are mapped to each other under P_α so that the cut C_{34} is invariant. Thus we can focus on the first term only. In the first term, multiply numerator and denominator by $\langle \ell_1\, \ell_2 \rangle$ and perform the t integral to get

$$C_{34}^{(1)} = -\frac{\langle 3|P_{12}|6]^2 P_{34}^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 5|P_{61}|2]P_{612}^2} \int \langle \lambda_{\ell_1}\ d\lambda_{\ell_1}\rangle[\lambda_{\widetilde{\ell_1}}\ d\lambda_{\widetilde{\ell_1}}] \frac{\langle \ell_1|P_{12}|6]\langle \ell_1\ 3\rangle}{\langle \ell_1\ 5\rangle\langle \ell_1\ 4\rangle} \frac{1}{\langle \ell_1|P_{34}|\ell_1]^2}$$

(4.3)

We can rewrite this as

$$C_{34}^{(1)} = C \int \langle \lambda_{\ell_1}\ d\lambda_{\ell_1}\rangle[d\widetilde{\lambda}_{\ell_1}\ \partial_{\widetilde{\lambda}_{\ell_1}}] \left(\frac{\langle \ell_1|P_{12}|6]\langle \ell_1\ 3\rangle}{\langle \ell_1\ 5\rangle\langle \ell_1\ 4\rangle} \frac{[\eta\ \ell_1]}{\langle \ell_1|P_{34}|\ell_1]\langle \ell_1|P_{34}|\eta]} \right),$$

(4.4)

where we have defined the constant

$$C = -\frac{\langle 3|P_{12}|6]^2 P_{34}^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 5|P_{61}|2]P_{612}^2}.$$

At this stage, we have three poles: $|\ell_1\rangle = |4\rangle$, $|\ell_1\rangle = |5\rangle$, and $|\ell_1\rangle = |P_{23}|\eta]$. However, since $|\eta]$ is an arbitrary spinor, we can choose it to be $|\eta] = |4]$. It is easy to see that after making the above choice we reduce the integrand into

$$C_{34}^{(1)} = C \int \langle \lambda_{\ell_1}\ d\lambda_{\ell_1}\rangle[d\widetilde{\lambda}_{\ell_1}\ \partial_{\widetilde{\lambda}_{\ell_1}}] \left(\frac{\langle \ell_1|P_{12}|6][4\ \ell_1]}{[3\ 4]\langle \ell_1\ 5\rangle\langle \ell_1\ 4\rangle\langle \ell_1|P_{34}|\ell_1]} \right)$$

(4.5)

where only one pole $|\ell_1\rangle = |5\rangle$ gives nonzero contribution (for the pole $|\ell_1\rangle = |4\rangle$, since the factor $[4\ \ell_1]$ appears in the numerator, the residue is zero). Reading out the residue we get

$$C_{34}^{(1)} = -\frac{[4\ 5]\langle 5|P_{345}|6]\langle 3|P_{345}|6]^2}{\langle 4\ 5\rangle[6\ 1][1\ 2]\langle 5|P_{345}|5]\langle 5|P_{345}|2]P_{345}^2}$$

(4.6)

Note here that the result of the integration is the residue with an extra minus sign. However, the coefficient of a bubble will be just the sum of the residues at the poles.

The Results

Since for the other integrations, the procedure is exactly same as above, we list our results directly. The coefficient in the cut C_{612} is given by

$$c_{2:3;6} = -\frac{\langle 3|P_{612}|6]^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5|P_{612}|2]P_{612}^2}\left(\frac{\langle 3|P_5|6]P_{612}^2}{\langle 5|P_{612}|5]} + \frac{\langle 3|P_{612}P_2P_{612}|6]}{\langle 2|P_{612}|2]}\right) \quad (4.7)$$

The coefficient in the cut C_{234} is given by

$$c_{2:3;2} = -\frac{\langle 1|P_{561}|4]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[2\ 3][3\ 4]\langle 5|P_{561}|2]P_{561}^2}\left(\frac{\langle 1|P_2|4]P_{561}^2}{\langle 2|P_{561}|2]} + \frac{\langle 1|P_{561}P_5P_{561}|4]}{\langle 5|P_{561}|5]}\right) \quad (4.8)$$

The coefficient in the cut C_{34} is given by

$$c_{2:2;3} = \frac{[4\ 5]\langle 5|P_{345}|6]\langle 3|P_{345}|6]^2}{\langle 4\ 5\rangle[6\ 1][1\ 2]\langle 5|P_{345}|5]\langle 5|P_{345}|2]P_{345}^2} + \frac{\langle 2\ 3\rangle\langle 1|P_{234}|2]\langle 1|P_{234}|4]^2}{[2\ 3]\langle 5\ 6\rangle\langle 6\ 1\rangle\langle 2|P_{234}|2]\langle 5|P_{234}|2]P_{234}^2} \quad (4.9)$$

Finally, the coefficient in the cut C_{61} is given by

$$c_{2:2;6} = \frac{[5\ 6]\langle 5|P_{561}|4]\langle 1|P_{561}|4]^2}{\langle 5\ 6\rangle[2\ 3][3\ 4]\langle 5|P_{561}|2]\langle 5|P_{561}|5]P_{561}^2} + \frac{\langle 1\ 2\rangle\langle 3|P_{612}|2]\langle 3|P_{612}|6]^2}{[1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5|P_{612}|5]\langle 2|P_{612}|2]P_{612}^2} \quad (4.10)$$

It is easy to check that the sum of all these coefficients is equal to the tree-level amplitude as required by the divergent behavior discussed in Section 2.

Comparison with Known Results:

The same amplitude has been calculated in [22], where the result was given by

$$\begin{aligned}A =&\, a_1\mathrm{K}_0[s_{61}] + a_2\mathrm{K}_0[s_{34}] - \frac{1}{2}\left[b_1\frac{\mathrm{L}_0[s_{345}/s_{61}]}{s_{61}} + b_2\frac{\mathrm{L}_0[s_{234}/s_{34}]}{s_{34}}\right.\\ &\left.+b_3\frac{\mathrm{L}_0[s_{234}/s_{61}]}{s_{61}} + b_4\frac{\mathrm{L}_0[s_{345}/s_{34}]}{s_{34}}\right]\end{aligned} \quad (4.11)$$

with

$$\begin{aligned}a_1 =&\, a_2 = \frac{1}{2}A^{\text{tree}}\\ b_1 =&\, \frac{\langle 3|P_{345}|6]^2\langle 3|P_{345}P_{345}2 - P_{345}2P_{345}|6]}{\langle 5|P_{345}|2][6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P_{345}^2}\\ b_4 =&\, \frac{\langle 3|P_{345}|6]^2\langle 3|-5P_{345}P_{345} + P_{345}5P_{345}|6]}{\langle 5|P_{345}|2][6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P_{345}^2}\\ b_2 =&\, \frac{\langle 1|P_{234}|4]^2\langle 1|P_{234}2P_{234} - P_{234}P_{234}2|4]}{\langle 5|P_{234}|2][2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2}\\ b_3 =&\, \frac{\langle 1|P_{234}|4]^2\langle 1|5P_{234}P_{234} - P_{234}5P_{234}|4]}{\langle 5|P_{234}|2][2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2}\end{aligned} \quad (4.12)$$

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The functions K_0 and L_0 are defined by [3]

$$K_0(s) = \frac{1}{\epsilon(1-2\epsilon)}(-s)^{-\epsilon}, \qquad L_0(r) = \frac{\ln(r)}{1-r}. \tag{4.13}$$

The function K_0 is proportional to the bubble integral, and the function L_0 is related to the Feynman parameter integral for a two-mass triangle integral. These two functions are in fact related by the identity

$$\frac{L_0[s_1/s_2]}{s_2} = \frac{K_0[s_2] - K_0[s_1]}{s_2 - s_1} + \mathcal{O}(\epsilon). \tag{4.14}$$

Therefore (4.11) can be brought to the following form:

$$A = \left(a_1 - \frac{1}{2}\frac{b_1}{s_{61} - s_{345}} - \frac{1}{2}\frac{b_3}{s_{61} - s_{234}}\right)K_0[s_{61}] + \left(a_2 - \frac{1}{2}\frac{b_2}{s_{34} - s_{234}} - \frac{1}{2}\frac{b_4}{s_{34} - s_{345}}\right)K_0[s_{34}]$$

$$+ \frac{1}{2}\left[\left(\frac{b_1}{s_{61} - s_{345}} + \frac{b_4}{s_{34} - s_{345}}\right)K_0[s_{345}] + \left(\frac{b_2}{s_{34} - s_{234}} + \frac{b_3}{s_{61} - s_{234}}\right)K_0[s_{234}]\right] \tag{4.15}$$

Each quantity in parentheses corresponds to one of the bubble coefficients in (4.7)-(4.10). It is easy to check that our results agree with (4.15).

4.2. Second Configuration: $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$

This configuration has the following $\mathbb{Z}_2$ symmetry: $P_\alpha : i \leftrightarrow 7 - i$ plus conjugation. With this helicity assignment, we have box, triangle and bubble contributions. The box part is easy to calculate by quadruple cuts.

The nonzero box contributions come from both two-mass hard and one-mass box integrals. For the two-mass hard box integrals $I_{4:2;i}^{2m\ h}$, $i = 2, 4, 6$, the coefficients are

$$c_{4:2;2}^{2m\ h} = -\frac{P_{61}^2 P_{456}^2}{2}\frac{\langle 4|P_{456}|3]^2 \langle 4\ 6\rangle[3\ 1]}{\langle 6|P_{456}|1]^2 \langle 4\ 5\rangle\langle 5\ 6\rangle[1\ 2][2\ 3]}$$

$$c_{4:2;4}^{2m\ h} = -\frac{P_{23}^2 P_{612}^2}{2}\frac{[2\ 6]\langle 4|P_{612}|2]\langle 4|P_{612}|6]^2}{\langle 4\ 5\rangle[6\ 1][1\ 2]\langle 5|P_{612}|2]\langle 3|P_{612}|2]^2} \tag{4.16}$$

$$c_{4:2;6}^{2m\ h} = -\frac{P_{45}^2 P_{561}^2}{2}\frac{\langle 1\ 5\rangle\langle 1|P_{561}|3]^2 \langle 5|P_{561}|3]}{[2\ 3]\langle 5\ 6\rangle\langle 6\ 1\rangle\langle 5|P_{561}|2]\langle 5|P_{561}|4]^2}$$

It is easy to see that $c_{4:2;4}^{2m\ h}$ and $c_{4:2;6}^{2m\ h}$ are mapped to each other under P_α while $c_{4:2;2}^{2m\ h}$ is invariant. For the one-mass box integrals $I_{4;i}^{1m}$, $i = 5, 6$, the coefficients are

$$c_{4;5}^{1m} = -\frac{P_{23}^2 P_{34}^2}{2}\frac{\langle 1|P_{234}|2]\langle 1|P_{234}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[4\ 2]^2\langle 5|P_{234}|2]P_{234}^2}$$

$$c_{4;6}^{1m} = -\frac{P_{34}^2 P_{45}^2}{2}\frac{\langle 5|P_{345}|6]\langle 4|P_{345}|6]^2}{[6\ 1][1\ 2]\langle 3\ 5\rangle^2\langle 5|P_{345}|2]P_{345}^2} \tag{4.17}$$

which are mapped to each other under P_α.

For the triangle part, as we argued in general, we need only pay attention to the three-mass triangle part, I_3^{3m}. For this case, there is only one I_3^{3m} function with the distribution $(23|45|61)$. We can calculate the coefficient by triple cut in principle, but we choose to read it out by a corresponding double cut integration which we will evaluate presently.

For the bubble part, we have following cuts: three particle channels C_{123}, C_{612} and C_{234}; two particle channels C_{23}, C_{34}, C_{45} and C_{61}. Among them, the pairs (C_{612}, C_{234}) and (C_{23}, C_{45}) are exchanged under $\mathbb{Z}_2$ symmetry while others are invariant. So in total we have five independent double cuts with the following integrands.

$$C_{123} = -\int d\mu \frac{\langle 4|P_{456}|3]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{456}^2} \frac{[3\ \ell_1]\langle \ell_1\ 4\rangle}{[1\ \ell_1]\langle \ell_1\ 6\rangle}$$

$$C_{234} = -\int d\mu \frac{\langle 1|P_{561}|3]^2}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{561}^2} \frac{[3\ \ell_2]\langle \ell_2\ 1\rangle[3\ \ell_1]}{[4\ \ell_2]\langle \ell_2\ 5\rangle[2\ \ell_1]}$$

$$C_{23} = \int d\mu \left(\frac{[5\ 6]^4\langle 4\ 2\rangle^2}{\langle 2\ 3\rangle[5\ 6][6\ 1]P_{561}^2\langle 4|P_{561}|1]} \frac{\langle 4\ \ell_1\rangle[\ell_1\ 3]}{[2\ \ell_1]\langle \ell_1|P_{561}|5]} \right.$$
$$- \frac{\langle 4|P_{456}|3]^2}{[2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{456}^2\langle 4|P_{456}|1]} \frac{\langle 4|P_{456}|\ell_2]\langle \ell_2\ 2\rangle\langle \ell_2|P_{23}P_{456}|4\rangle}{\langle 6|P_{456}|\ell_2]\langle \ell_2\ 3\rangle\langle \ell_2|P_{23}|1]}$$
$$\left. + \frac{1}{[4\ 5]\langle 6\ 1\rangle\langle 2\ 3\rangle} \frac{\langle 1|\ell_2+4+5|5]}{(\ell_2+4+5)^2} \frac{[3\ \ell_1]}{[2\ \ell_1]} \frac{\langle \ell_1|P_{23}|5]}{\langle \ell_1|P_{23}|4]} \frac{(\langle 1|6|5]\langle 2\ \ell_1\rangle + \langle 2|3|5]\langle 1\ \ell_1\rangle)^2}{\langle \ell_1|P_{61}|5]\langle 6|P_{45}P_{23}|\ell_1\rangle} \right)$$

$$C_{34} = \int d\mu \left(\frac{[3\ 4]\langle 4|P_{612}|6]^2}{[6\ 1][1\ 2]P_{34}^2P_{612}^2\langle 5|P_{612}|2]} \frac{[3\ \ell_2]\langle \ell_2|P_{612}|6]}{[4\ \ell_2]\langle \ell_2\ 5\rangle} \right.$$
$$\left. + \frac{\langle 3\ 4\rangle\langle 1|P_{561}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle P_{34}^2P_{561}^2\langle 5|P_{561}|2]} \frac{\langle \ell_1\ 4\rangle\langle 1|P_{561}|\ell_1]}{\langle \ell_1\ 3\rangle[2\ \ell_1]} \right)$$

$$C_{61} = \int d\mu \left(\frac{[1\ 6]\langle 1\ 2\rangle^2[3\ 5]^4}{[3\ 4][4\ 5]\langle 2|P_{345}|5]P_{61}^2P_{345}^2} \frac{[6\ \ell_2]\langle 2\ \ell_1\rangle}{[1\ \ell_2]\langle \ell_1|P_{345}|3]} \right.$$
$$+ \frac{\langle 6\ 1\rangle[5\ 6]^2\langle 4\ 2\rangle^4}{\langle 2\ 3\rangle\langle 3\ 4\rangle\langle 2|P_{234}|5]P_{61}^2P_{234}^2} \frac{\langle \ell_1\ 1\rangle[5\ \ell_2]}{\langle \ell_1\ 6\rangle\langle 4|P_{234}|\ell_2]}$$
$$\left. + \frac{1}{\langle 6\ 1\rangle[2\ 3]\langle 4\ 5\rangle} \frac{\langle 4|\ell_2+2+3|3]}{(\ell_2+2+3)^2} \frac{\langle \ell_1|P_{61}|3]\langle 4\ \ell_1\rangle\langle 1\ \ell_1\rangle(\langle 4|P_{12}|3]\langle 1\ \ell_1\rangle - \langle 4\ 1\rangle\langle \ell_1|6|3])^2}{\langle \ell_1|P_{61}P_{23}|4\rangle\langle \ell_1\ 5\rangle\langle \ell_1\ 6\rangle\langle \ell_1|P_{61}|2]\langle \ell_1|P_{45}|3]} \right)$$
$$\tag{4.18}$$

Again, when we calculate integrands we need to use the tree-level amplitude of four gluons and a pair of fermions or scalars given in Appendix B.

Now we will perform the integration. Compared to the previous subsection, this integration is more involved. We use two examples to demonstrate our method.

The Cut C_{234}

The integral in (4.18) may be reduced to

$$C_{234} = C \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \frac{P_{234}^2}{\langle \ell | P_{234} | \ell]^2} \frac{[3 \, \ell]}{[4 \, \ell]} \frac{\langle \ell \, 1 \rangle \langle \ell | P_{234} | 3]}{\langle \ell \, 5 \rangle \langle \ell | P_{234} | 2]} \tag{4.19}$$

after the t integration, where $\ell = \ell_2$ and

$$C = -\frac{\langle 1 | P_{234} | 3]^2}{[2 \, 3][3 \, 4] \langle 5 \, 6 \rangle \langle 6 \, 1 \rangle P_{234}^2}.$$

Now we can see the new feature in the above integrand: it depends on the antiholomorphic variable $|\ell]$ as well as the holomorphic variable $|\ell\rangle$. To simplify the calculation, we split the above integrand by multiplying numerator and denominator by $\langle \ell | P_{234} | 4]$. Then use a Schouten identity to rewrite $[3 \, \ell] \langle \ell | P_{234} | 4] = [3 \, 4] \langle \ell | P_{234} | \ell] - [3 | P_{234} | \ell \rangle [\ell \, 4]$, we get

$$\begin{aligned} C_{234} &= C P_{234}^2 \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \frac{\langle \ell \, 1 \rangle \langle \ell | P_{234} | 3]}{\langle \ell \, 5 \rangle \langle \ell | P_{234} | 2] \langle \ell | P_{234} | 4]} \left(\frac{[3 \, 4]}{[4 \, \ell] \langle \ell | P_{234} | \ell]} + \frac{\langle \ell | P_{234} | 3]}{\langle \ell | P_{234} | \ell]^2} \right) \\ &\equiv C_{234}^{(1)} + C_{234}^{(2)} \end{aligned} \tag{4.20}$$

Splitting the whole integral into two pieces not only simplifies the calculation but also provides a nice way to separate the various contributions. As we will see shortly, the first term will produce a pure logarithmic contribution which is related to the imaginary part of the box integral while the second term produces a rational function which is exactly the coefficient of the bubble function. This same pattern will show up in every calculation we meet. Since we have already found the coefficient of the box integral from a quadruple cut, we can neglect this term and concentrate on the rational piece only. In other words, this splitting is the canonical way to separate box, triangle and bubble contributions.

Now let us do the integration term by term. For $C_{234}^{(2)}$ we have

$$\begin{aligned} C_{234}^{(2)} &= C P_{234}^2 \int \langle \ell \, d\ell \rangle [\ell \, d\ell] \frac{\langle \ell \, 1 \rangle \langle \ell | P_{234} | 3]}{\langle \ell \, 5 \rangle \langle \ell | P_{234} | 2] \langle \ell | P_{234} | 4]} \frac{\langle \ell | P_{234} | 3]}{\langle \ell | P_{234} | \ell]^2} \\ &= - C P_{234}^2 \left(\frac{\langle \ell \, 1 \rangle \langle \ell | P_{234} | 3]^2 [\eta \, \ell]}{\langle \ell \, 5 \rangle \langle \ell | P_{234} | 2] \langle \ell | P_{234} | 4] \langle \ell | P_{234} | \eta] \langle \ell | P_{234} | \ell]} \right)_{\text{pole}} \end{aligned} \tag{4.21}$$

where $(\ )_{\text{pole}}$ means to sum the residues of all poles, which is the consequence of the integration $\int \langle \ell \, d\ell \rangle [\ell \, d\ell]$. Choosing $|\eta] = |P_{234} | 1 \rangle$, we find that there are three poles giving

nontrivial contributions: $|\ell\rangle = |5\rangle$, $|\ell\rangle = |P_{234}|2]$ and $|\ell\rangle = |P_{234}|4]$. Summing up these contributions we finally get

$$
\begin{aligned}
C_{234}^{(2)} =& \frac{\langle 1|P_{234}|3]^2}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2} \left(-\frac{\langle 1|P_6|5]\langle 5|P_{234}|3]^2}{\langle 5|P_{234}|2]\langle 5|P_{234}|4]\langle 5|P_{234}|5]} \right. \\
&\left. +\frac{\langle 1\ 2\rangle[2\ 3]^2 P_{234}^2}{[2\ 4]\langle 5|P_{234}|2]\langle 2|P_{234}|2]} + \frac{\langle 1\ 4\rangle[3\ 4]^2 P_{234}^2}{[4\ 2]\langle 5|P_{234}|4]\langle 4|P_{234}|4]} \right)
\end{aligned}
\tag{4.22}
$$

For $C_{234}^{(1)}$ we have

$$
\begin{aligned}
C_{234}^{(1)} =& C P_{234}^2 \int \langle \ell\ d\ell\rangle [\ell\ d\ell] \frac{\langle \ell\ 1\rangle\langle \ell|P_{234}|3]}{\langle \ell\ 5\rangle\langle \ell|P_{234}|2]\langle \ell|P_{234}|4]} \frac{[3\ 4]\langle \ell\ 4\rangle}{\langle \ell|p_4|\ell]\langle \ell|P_{234}|\ell]} \\
=& C P_{234}^2 \int_0^1 dz \int \langle \ell\ d\ell\rangle [\ell\ d\ell] \frac{\langle \ell\ 1\rangle\langle \ell|P_{234}|3][3\ 4]\langle \ell\ 4\rangle}{\langle \ell\ 5\rangle\langle \ell|P_{234}|2]\langle \ell|P_{234}|4]} \frac{1}{\langle \ell|P|\ell]^2},
\end{aligned}
\tag{4.23}
$$

where in the second line we have used the Feynman parametrization to rewrite the integrand with $P = z P_{4561} - p_4$. At this stage, the integration is easy to do and given by

$$
\begin{aligned}
C_{234}^{(1)} =& -C P_{234}^2 \int_0^1 dz \left(\frac{\langle \ell|P_{234}|3][3\ 4]\langle \ell\ 1\rangle}{\langle \ell\ 5\rangle\langle \ell|P_{234}|2]\langle \ell|P_{234}|4]} \frac{\langle 4|P|\ell]}{P^2\langle \ell|P|\ell]} \right) |_{\text{pole}} \\
=& C P_{234}^2 \int_0^1 dz \left(\frac{\langle 5|P_{234}|3][3\ 4]\langle 1\ 5\rangle}{\langle 5|P_{234}|2]\langle 5|P_{234}|4]} \frac{\langle 4|P|5]}{P^2\langle 5|P|5]} + \frac{[2\ 3][3\ 4]\langle 1|P_{234}|2]}{\langle 5|P_{234}|2][2\ 4]} \frac{\langle 4|P P_{234}|2]}{P^2\langle 2|P_{234}P P_{234}|2]} \right. \\
& \left. -\frac{[3\ 4]^2\langle 1|P_{234}|4]}{[4\ 2]\langle 5|P_{234}|4]} \frac{\langle 4|P P_{234}|4]}{P^2\langle 4|P_{234}P P_{234}|4]} \right) \\
=& -\frac{\langle 1\ 5\rangle\langle 1|P_{234}|3]^2\langle 5|P_{234}|3]}{[2\ 3]\langle 5\ 6\rangle\langle 6\ 1\rangle\langle 5|P_{234}|2]\langle 5|P_{234}|4]^2} \left(\ln\frac{P_{234}^2}{P_{234}^2 - P_{23}^2} + \ln\frac{-P_{45}^2}{P_{234}^2 - P_{61}^2} \right) \\
& +\frac{\langle 1|P_{234}|2]\langle 1|P_{234}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[4\ 2]^2\langle 5|P_{234}|2] P_{234}^2} \ln\left(\frac{P_{23}^2 P_{34}^2}{(P_{234}^2 - P_{23}^2)(P_{234}^2 - P_{34}^2)} \right)
\end{aligned}
\tag{4.24}
$$

It is easy to see that the first logarithm is the contribution of two-mass-hard box integral $I_{4:2;6}^{2m\ h}$ and the second logarithm is the contribution of the one-mass box function $I_{4;5}^{1m}$.

The Cut C_{23}

Now we consider the cut C_{23} from which we can read out the coefficient of triangle function. The integrand is given by three terms. For the first two terms, the calculations are similar to those above, so we just list the results for coefficients of bubbles (and neglect

those of boxes). They are

$$
c_{2:2;2} = c_{2:2;2}^{(1)} + c_{2:2;2}^{(2)} + c_{2:2;2}^{(3)};
$$

$$
c_{2:2;2}^{(1)} = \frac{\langle 2\,4 \rangle^2 [5\,6]^4}{[5\,6][6\,1]P_{561}^2} \frac{\langle 4|P_{561}|5]}{\langle 4|P_{561}|1]} \frac{\langle 5|P_{561}|3]}{\langle 3|P_{561}|5]} \frac{1}{\langle 5|P_{561}P_{23}P_{561}|5]}
$$

$$
c_{2:2;2}^{(2)} = \frac{\langle 4|P_{456}|3]^2}{[2\,3]\langle 4\,5 \rangle \langle 5\,6 \rangle P_{456}^2 \langle 4|P_{456}|1]} \left(-\frac{[3\,2]\langle 2\,1 \rangle \langle 4|P_{456}|1]^2}{[2\,1]\langle 6|P_{456}|1]\langle 1|P_{23}|1]} \right.
$$

$$
\left. + \frac{\langle 4\,6 \rangle^2 [2\,3]\langle 2|P_{456}|6](P_{456}^2)^2}{\langle 6|P_{456}|2]\langle 6|P_{456}|1]\langle 6|P_{456}P_{23}P_{456}|6]} \right)
$$

(4.25)

The third term is a little involved. Defining

$$
g(\ell) = -\frac{\langle \ell|P_{23}|5]}{\langle \ell|P_{23}|4]} \frac{(\langle 1|6|5]\langle 2\,\ell \rangle + \langle 2|3|5]\langle 1\,\ell \rangle)^2}{\langle \ell|P_{61}|5]\langle \ell|P_{23}P_{45}|6]}
$$

$$
Q = \frac{P_{61}^2 P_{23} + P_{23}^2 P_{61}}{P_{23}^2}, \qquad Q^2 = \frac{P_{61}^2 P_{45}^2}{P_{23}^2}
$$

(4.26)

it is easy to see that the third integration becomes (with $C = \frac{1}{[4\,5]\langle 6\,1 \rangle \langle 2\,3 \rangle}$)

$$
C_{23}^{(3)} = -C \int \langle \ell\,d\ell \rangle [\ell\,d\ell] \frac{P_{23}^2}{\langle \ell|P_{23}|\ell]^2} \frac{[3\,\ell]}{[2\,\ell]} \frac{\langle 1|P_{61}|5]\langle \ell|P_{23}|\ell] + P_{23}^2\langle 1|\ell|5]}{\langle \ell|Q|\ell]P_{23}^2} g(\ell)
$$

(4.27)

after the t-integration. Now we can use our method to split the antiholomorphic part in the denominator as

$$
C_{23}^{(3)} = C \int \langle \ell\,d\ell \rangle [\ell\,d\ell] \left[\frac{1}{\langle \ell|P_{23}|\ell][2\,\ell]} \frac{g(\ell)[3\,2]}{\langle \ell|Q|2]} \left(-\langle 1|6|5] + \frac{\langle 3\,2 \rangle \langle 1\,\ell \rangle [2\,5]}{\langle \ell\,3 \rangle} \right) \right.
$$

$$
- \frac{1}{\langle \ell|P_{23}|\ell]^2} \frac{g(\ell)P_{23}^2\langle 1\,\ell \rangle \langle \ell|P_{23}|5]\langle \ell\,2 \rangle}{\langle \ell\,3 \rangle \langle \ell|P_{23}Q|\ell \rangle}
$$

(4.28)

$$
\left. - \frac{1}{\langle \ell|P_{23}|\ell]\langle \ell|Q|\ell]} \frac{g(\ell)\langle \ell|Q|3]}{\langle \ell|Q|2]\langle \ell|P_{23}Q|\ell \rangle} \left(\langle 1|6|5]\langle \ell|P_{23}Q|\ell \rangle + P_{23}^2\langle 1\,\ell \rangle \langle \ell|Q|5] \right) \right]
$$

The first, second and third lines of (4.28) are respectively the contributions from the box, bubble, and triangle. For the second term, it is easy to read out

$$
c_{2:2;2}^{(3)} = -\frac{1}{[4\,5]\langle 6\,1 \rangle} \left(\frac{g(\ell)\langle 1\,\ell \rangle \langle \ell|P_{23}|5][3\,\ell]}{\langle \ell\,3 \rangle \langle \ell|P_{23}Q|\ell \rangle \langle \ell|P_{23}|\ell]} \right)_{\text{pole}}
$$

(4.29)

There are five poles giving non-zero contributions: three from $g(\ell)$ and two from the factor $\langle \ell|P_{23}Q|\ell \rangle$. To find the location of the last two poles, we use the following method. Taking

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two arbitrary external momenta $|a\rangle$ and $|b\rangle$, since the spinor is two dimensional, we can represent $|\ell\rangle$ as

$$|\ell\rangle = (|a\rangle + x|b\rangle) \qquad (4.30)$$

with undetermined complex variable x.[5] Putting it back into $\langle \ell|P_{23}Q|\ell\rangle$, we get a quadratic equation whose solutions are

$$x_{\pm} = \frac{-(\langle a|P_{23}Q|b\rangle + \langle b|P_{23}Q|a\rangle) \pm \langle a\ b\rangle\sqrt{\Delta_{3m}}}{2\langle b|P_{23}Q|b\rangle} \qquad (4.31)$$

where

$$\Delta_{3m} = (P_{61}^2)^2 + (P_{45}^2)^2 + (P_{23}^2)^2 - 2P_{61}^2 P_{23}^2 - 2P_{23}^2 P_{45}^2 - 2P_{45}^2 P_{61}^2 \qquad (4.32)$$

Now we can write the residue as

$$c_{2:2;2}^{(3)} = -\frac{1}{[4\ 5]\langle 6\ 1\rangle} \sum_{i=1}^{5} \left(\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle 1\ \ell\rangle\langle\ell|P_{23}|5][3\ \ell]}{\langle\ell\ 3\rangle\langle\ell|P_{23}Q|\ell\rangle\langle\ell|P_{23}|\ell]} \right)_{\ell \to \ell_i} \qquad (4.33)$$

with the following five poles:

$$|\ell_1\rangle = |P_{23}|4], \quad |\ell_2\rangle = |P_{61}|5], \quad |\ell_3\rangle = |P_{23}P_{45}|6\rangle$$
$$|\ell_4\rangle = |a\rangle + x_+|b\rangle, \quad |\ell_5\rangle = |a\rangle + x_-|b\rangle, \qquad (4.34)$$

It does not seem useful to write the expression (4.33) explicitly because it is rather long and complicated. However, its structure is very clear and easy to implement in a computer program. One has to evaluate the right hand side of (4.33) for the five values of $|\ell\rangle$ given by (4.34) and add up the obtained contributions. We find that the (4.33) is the most convenient way to present the answer.

It is worth remarking that although the square root shows up in above expression, the final result $c_{2:2;2}^{(3)}$ is rational. A similar feature is encountered in calculating the coefficient of a four-mass box integral from a quadruple cut [5].

Now we are left with the third term of $C_{23}^{(3)}$ which can be expressed as

$$C_{23}^{(3;3)} = -C \int_0^1 dz \int \langle \ell\ d\ell\rangle[\ell\ d\ell] \frac{1}{\langle\ell|P|\ell]^2} \frac{g(\ell)\langle\ell|Q|3]}{\langle\ell|Q|2]\langle\ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle\ell|P_{23}Q|\ell\rangle + P_{23}^2\langle 1\ \ell\rangle\langle\ell|Q|5] \right) \qquad (4.35)$$

---

[5] In principle we should write $|\ell\rangle = \alpha(|a\rangle + x|b\rangle)$, but one can check that the factor α drops out of the final expression, so we can set $\alpha = 1$.

with $P \equiv (1-z)P_{23} + zQ$. Do the integration $\int \langle \ell\, d\ell \rangle [\ell\, d\ell]$, and we are left with

$$
\begin{aligned}
C_{23}^{(3;3)} =& C \int_0^1 dz \left[\frac{[\eta\, \ell]}{\langle \ell|P|\ell]\langle \ell|P|\eta]} \frac{g(\ell)\langle \ell|Q|3]}{\langle \ell|Q|2]\langle \ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle \ell|P_{23}Q|\ell\rangle + P_{23}^2 \langle 1\ \ell\rangle\langle \ell|Q|5] \right) \right]_{\text{pole}} \\
=& C \int_0^1 dz \left[\frac{\langle \widetilde{\eta}|P|\ell]}{\langle \ell|P|\ell] P^2} \frac{g(\ell)\langle \ell|Q|3]}{\langle \ell\ \widetilde{\eta}\rangle\langle \ell|Q|2]\langle \ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle \ell|P_{23}Q|\ell\rangle + P_{23}^2 \langle 1\ \ell\rangle\langle \ell|Q|5] \right) \right]_{\text{pole}}
\end{aligned}
\tag{4.36}
$$

where at the second line we have chosen $|\eta] = |P|\widetilde{\eta}\rangle$ with arbitrary $\widetilde{\eta}$. The advantage of this choice is that now every pole is independent of the Feynman parameter z, and we can evaluate $\int_0^1 dz$ before taking residues of poles. The integration is of the following pattern,

$$
\begin{aligned}
I_f \equiv & \int_0^1 dz \frac{(zc_1 + c_2)}{(a_0 z^2 + a_1 z + a_2)(zb_1 + b_2)} \\
=& \int_0^1 dz \frac{b_1(-b_2 c_1 + b_1 c_2)}{(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{1}{(zb_1 + b_2)} \\
& + \int_0^1 dz \frac{(b_2 c_1 - b_1 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{(2za_0 + a_1)}{(a_0 z^2 + a_1 z + a_2)} \\
& + \int_0^1 dz \frac{(2a_2 b_1 c_1 - a_1 b_2 c_1 - a_1 b_1 c_2 + 2a_0 b_2 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)} \frac{1}{(a_0 z^2 + a_1 z + a_2)}
\end{aligned}
\tag{4.37}
$$

where

$$
\begin{aligned}
a_0 =& (Q - P_{23})^2, \qquad a_1 = 2P_{23} \cdot (Q - P_{23}), \qquad a_2 = P_{23}^2 \\
b_1 =& \langle \ell|(Q - P_{23})|\ell], \qquad b_2 = \langle \ell|P_{23}|\ell] \qquad c_1 = \langle \widetilde{\eta}|(Q - P_{23})|\ell], \qquad c_2 = \langle \widetilde{\eta}|P_{23}|\ell]
\end{aligned}
\tag{4.38}
$$

We have split I_f into three terms. Among them, the first two terms give the imaginary part of the box integral while the last term is exactly the cut contribution of the three-mass triangle function. Let us define the function

$$
R_1(a_j, b_j, c_j) = \frac{(2a_2 b_1 c_1 - a_1 b_2 c_1 - a_1 b_1 c_2 + 2a_0 b_2 c_2)}{2(a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2)}.
\tag{4.39}
$$

We would like to point out that there is an important subtlety with the above procedure.[6] It is correct only if the denominator $a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2$ does not vanish. However, it can be shown that this denominator vanishes for $|\ell\rangle$ satisfying $\langle \ell|P_{23}Q|\ell\rangle = 0$ which is one

---

[6] This subtlety was overlooked in previous versions of this paper. We would like to thank P. Mastrolia for stimulating discussions pointing to this issue and R. K. Ellis who independently found the problem and informed us.

of the poles in eq. (4.35). This means that we have to redo the simple pole expansion assuming that $a_2 b_1^2 - a_1 b_1 b_2 + a_0 b_2^2 = 0$. In fact, we need only the term proportional to $\frac{1}{(a_0 z^2 + a_1 z + a_2)}$ because only such a term contributes to the three-mass triangle upon the z-integration. Redoing the simple fraction expansion we find that $\frac{1}{(a_0 z^2 + a_1 z + a_2)}$ is multiplied by the function $R_2(a_j, b_j, c_j)$, where

$$R_2(a_j, b_j, c_j) = \frac{(a_2 b_2 c_1 + a_2 b_1 c_2 - a_1 b_2 c_2)}{b_2(2a_2 b_1 - a_1 b_2)} \tag{4.40}$$

and all a's, b's and c's are as before. Thus, we finally have the coefficient of three-mass triangle as

$$
\begin{aligned}
c_{3:2:2;2}^{3m} = {} & \frac{1}{[4\ 5]\langle 6\ 1\rangle\langle 2\ 3\rangle} \times \\
& \sum_{i=1}^{5} \left[\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle\ell|Q|3]R_1(a_j,b_j,c_j)}{\langle\ell\ \widehat{\eta}\rangle\langle\ell|Q|2]\langle\ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle\ell|P_{23}Q|\ell\rangle + P_{23}^2\langle 1\ \ell\rangle\langle\ell|Q|5] \right) \right]_{\ell\to\ell_i} \\
& + \frac{1}{[4\ 5]\langle 6\ 1\rangle\langle 2\ 3\rangle} \times \\
& \sum_{i=6}^{7} \left[\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle\ell|Q|3]R_2(a_j,b_j,c_j)}{\langle\ell\ \widehat{\eta}\rangle\langle\ell|Q|2]\langle\ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle\ell|P_{23}Q|\ell\rangle + P_{23}^2\langle 1\ \ell\rangle\langle\ell|Q|5] \right) \right]_{\ell\to\ell_i} .
\end{aligned}
\tag{4.41}
$$

Here, in the first term, the summation is over the poles

$$
\begin{aligned}
|\ell_1\rangle &= |P_{23}|4], \quad |\ell_2\rangle = |P_{61}|5], \quad |\ell_3\rangle = |P_{23}P_{45}|6\rangle, \\
|\ell_4\rangle &= |Q|2], \quad |\ell_5\rangle = |\widetilde{\eta}\rangle.
\end{aligned}
\tag{4.42}
$$

In the second term, the summation is over the remaining two poles

$$|\ell_6\rangle = |a\rangle + x_+|b\rangle, \qquad |\ell_7\rangle = |a\rangle + x_-|b\rangle. \tag{4.43}$$

We can choose $|\widetilde{\eta}\rangle$ properly to reduce the number of poles further, but we do not do so here. It was checked numerically that eq. (4.41) is indeed independent of $|\widetilde{\eta}\rangle$.

The Results

Now we list the coefficients of the amplitude $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$. The box co-efficients are given by (4.16) and (4.17). The three-mass triangle coefficient is given by

eq. (4.41). For bubbles, we have six coefficients. From cuts in three-particle channels we have

$$
\begin{aligned}
c_{2:3;2} = & -\frac{\langle 1|P_{234}|3]^2}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2}\left(\frac{\langle 1|6|5]\langle 5|P_{561}|3]^2}{\langle 5|P_{561}|2]\langle 5|P_{561}|4]\langle 5|P_{561}|5]}\right.\\
& \left.+\frac{\langle 1\ 2\rangle[2\ 3]^2 P_{561}^2}{[2\ 4]\langle 5|P_{561}|2]\langle 2|P_{561}|2]}+\frac{\langle 1\ 4\rangle[3\ 4]^2 P_{561}^2}{[4\ 2]\langle 5|P_{561}|4]\langle 4|P_{561}|4]}\right)\\
c_{2:3;6} = & -\frac{\langle 4|P_{612}|6]^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P_{612}^2}\left(\frac{\langle 2|1|6]\langle 4|P_{612}|2]^2}{\langle 5|P_{612}|2]\langle 3|P_{612}|2]\langle 2|P_{612}|2]}\right.\\
& \left.+\frac{[5\ 6]\langle 4\ 5\rangle^2 P_{612}^2}{\langle 3\ 5\rangle\langle 5|P_{612}|2]\langle 5|P_{612}|5]}+\frac{[6\ 3]\langle 3\ 4\rangle^2 P_{612}^2}{\langle 3\ 5\rangle\langle 3|P_{612}|2]\langle 3|P_{612}|3]}\right)\\
c_{2:3;1} = & \ \frac{\langle 4|P_{456}|3]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle}\left(\frac{[6\ 3]\langle 6\ 4\rangle}{\langle 6|P_{123}|6]\langle 6|P_{123}|1]}-\frac{\langle 1|2|3]\langle 4|P_{123}|1]}{\langle 1|P_{123}|1]\langle 6|P_{123}|1]P_{123}^2}\right)
\end{aligned}
\tag{4.44}
$$

From cuts in two-particle channels, we have

$$
c_{2:2;3} = \frac{[3\ 5]\langle 5|P_{612}|6]\langle 4|P_{612}|6]^2}{[6\ 1][1\ 2]\langle 3\ 5\rangle\langle 5|P_{612}|2]\langle 5|P_{612}|5]P_{612}^2}+\frac{\langle 2\ 4\rangle\langle 1|P_{561}|2]\langle 1|P_{561}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[2\ 4]\langle 5|P_{561}|2]\langle 2|P_{561}|2]P_{561}^2}
\tag{4.45}
$$

and

$$
\begin{aligned}
c_{2:2;2} = & \ \frac{\langle 2\ 4\rangle^2[5\ 6]^4}{[5\ 6][6\ 1]P_{561}^2}\frac{\langle 4|P_{561}|5]}{\langle 4|P_{561}|1]}\frac{\langle 5|P_{561}|3]}{\langle 3|P_{561}|5]}\frac{1}{\langle 5|P_{561}P_{23}P_{561}|5]}\\
& +\frac{\langle 4|P_{456}|3]^2}{[2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{456}^2\langle 4|P_{456}|1]}\left(-\frac{[3\ 2]\langle 2\ 1\rangle\langle 4|P_{456}|1]^2}{[2\ 1]\langle 6|P_{456}|1]\langle 1|P_{23}|1]}\right.\\
& \left.+\frac{\langle 4\ 6\rangle^2[2\ 3]\langle 2|P_{456}|6](P_{456}^2)^2}{\langle 6|P_{456}|2]\langle 6|P_{456}|1]\langle 6|P_{456}P_{23}P_{456}|6]}\right)\\
& -\frac{1}{[4\ 5]\langle 6\ 1\rangle}\sum_{i=1}^{5}\left(\langle \ell\ \ell_i\rangle\frac{g(\ell)\langle 1\ \ell\rangle\langle \ell|P_{23}|5][3\ \ell]}{\langle \ell\ 3\rangle\langle \ell|P_{23}Q|\ell\rangle}\right)_{\ell\rightarrow\ell_i}
\end{aligned}
\tag{4.46}
$$

The coefficient $c_{2:2;4}$ can be obtained $c_{2:2;2}$ by a flip symmetry, while the coefficient $c_{2:2;6}$ can be expressed as $A_{\text{tree}}-\sum_{\text{others}}c_{2:r;i}$ from the divergence equation (2.8). We have obtained an analytic expression for $c_{2:2;6}$ using the techniques of this paper and checked numerically that (2.8) is satisfied. The analytic expression is rather long, so we omit it here for brevity.

4.3. Third Configuration: $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$

Now we move to the last helicity configuration $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$. It has the largest symmetry $\mathbb{Z}_6$ generated by $P_\alpha : i \rightarrow i+1$ plus conjugation. Because of this, we

need to calculate just one coefficient for each type of function and act on it by P_α to obtain all the others. The box coefficients are

$$c_{4:2;1}^{2m\ h} = \frac{P_{345}^2 P_{56}^2}{2} \frac{\langle 1|P_{345}|4]^2}{\langle 1\ 2\rangle[3\ 4]\langle 6|P_{345}|5]^2} \frac{\langle 1|P_{345}|5]\langle 6|P_{345}|4]}{\langle 2|P_{345}|5]\langle 6|P_{345}|3]} \tag{4.47}$$

and

$$c_{4;1}^{1m} = -\frac{P_{45}^2 P_{56}^2}{2} \frac{\langle 5|P_{456}|2]^2}{[1\ 2][2\ 3]\langle 4\ 6\rangle^2 P_{456}^2} \frac{\langle 4|P_{456}|2]\langle 6|P_{456}|2]}{\langle 4|P_{456}|1]\langle 6|P_{456}|3]} \tag{4.48}$$

For the triangle integrals, there are two two-mass triangles related by P_α. For the bubble integrals, the orbit of the cut in a three-particle channel contains three elements, while the orbit of the cut in a two-particle channel contains six elements. The representative integration we will perform is the following:

$$C_{123} = -\frac{\langle 5|P_{123}|2]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{123}^2} \frac{\langle 5|P_{123}|\ell_2]\langle \ell_2|P_{123}|2]}{\langle 6|P_{123}|\ell_2]\langle \ell_2|P_{123}|1]} \frac{[2\ \ell_2]\langle 5\ \ell_2\rangle}{[3\ \ell_2]\langle 4\ \ell_2\rangle}$$

$$C_{12} = \frac{[4\ 6]^4\langle 1\ 3\rangle^2[1\ 2]}{[4\ 5][5\ 6]\langle 3|P_{456}|6]P_{456}^2 P_{12}^2} \frac{[\ell_1\ 2]\langle \ell_1\ 3\rangle}{[\ell_1\ 1]\langle \ell_1|P_{456}|4]} + \frac{\langle 3\ 5\rangle^4\langle 1\ 2\rangle[2\ 6]^2}{\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 3|P_{345}|6]P_{345}^2 P_{12}^2} \frac{\langle 1\ \ell_2\rangle[6\ \ell_2]}{\langle 2\ \ell_2\rangle\langle 5|P_{345}|\ell_2]}$$

$$-\frac{1}{\langle 1\ 2\rangle[3\ 4]\langle 5\ 6\rangle} \frac{\langle 5|\ell_2+3+4|4]}{(\ell_2+3+4)^2} \frac{\langle 5\ \ell_1\rangle\langle 1|P_{12}|\ell_1]}{\langle 6\ \ell_1\rangle\langle 2|P_{12}|\ell_1]} \frac{\langle \ell_1|P_{12}|4]}{\langle 5|P_{34}P_{12}|\ell_1]} \frac{(\langle 5|6|4]\langle 1\ \ell_1\rangle + \langle 1|2|4]\langle 5\ \ell_1\rangle)^2}{\langle \ell_1|P_{56}|4]\langle \ell_1|P_{12}|3]}$$
$$\tag{4.49}$$

Now we perform the integration for these two cuts.

The cut C_{123}

After the t-integration and splitting we end up with

$$C_{123} = C\frac{P_{123}^2\langle 5|P_{123}|3]}{\langle 6|P_{123}|3]} \int \langle \ell\ d\ell\rangle[\ell\ d\ell] \frac{\langle \ell|P_{123}|2]\langle \ell\ 5\rangle}{\langle \ell|P_{123}|1]\langle \ell\ 4\rangle} \frac{1}{\langle \ell|P_{123}|3]} \left(\frac{\langle \ell|P_{123}|2]}{\langle \ell|P_{123}|\ell]^2} + \frac{[2\ 3]}{\langle \ell|P_{123}|\ell][3\ \ell]} \right)$$

$$+ C\frac{P_{123}^2\langle 5\ 6\rangle}{\langle 6|P_{123}|3]} \int \langle \ell\ d\ell\rangle[\ell\ d\ell] \frac{\langle \ell|P_{123}|2]\langle \ell\ 5\rangle}{\langle \ell|P_{123}|1]\langle \ell\ 4\rangle} \frac{1}{\langle \ell\ 6\rangle} \left(-\frac{\langle \ell|P_{123}|2]}{\langle \ell|P_{123}|\ell]^2} + \frac{\langle 6|P_{123}|2]}{\langle \ell|P_{123}|\ell]\langle 6|P_{123}|\ell]} \right)$$
$$\tag{4.50}$$

with $C = -\frac{\langle 5|P_{123}|2]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{123}^2}$. By our standard method, it is clear that among these four terms, the first and third terms will give rational functions and the second and fourth terms will give the logarithmic functions involved in the box contributions. Carrying out the integration for the first and third terms we read out the coefficient of bubble C_{123} as

$$c_{2:3;1} = -\frac{\langle 5|P_{123}|2]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{123}^2} \left(\frac{\langle 4|P_{123}|2]^2\langle 4\ 5\rangle\langle 5\ 6\rangle[4\ 6]}{\langle 4|P_{123}|1]\langle 4|P_{123}|3]\langle 4|P_{123}|4]\langle 4\ 6\rangle} \right.$$

$$-\frac{\langle 1\ 5\rangle[1\ 2]^2 P_{123}^2\langle 5|P_{123}|1]}{[1\ 3]\langle 4|P_{123}|1]\langle 6|P_{123}|1]\langle 1|P_{123}|1]} + \frac{\langle 5|P_{123}|3]}{\langle 6|P_{123}|3]} \frac{\langle 5\ 3\rangle[2\ 3]^2 P_{123}^2}{[3\ 1]\langle 4|P_{123}|3]\langle 3|P_{123}|3]}$$

$$\left. +\frac{\langle 5\ 6\rangle}{\langle 6|P_{123}|3]} \frac{\langle 5\ 4\rangle[4\ 6]\langle 6|P_{123}|2]^2}{\langle 6\ 4\rangle\langle 6|P_{123}|1]\langle 6|P_{123}|6]} \right) \tag{4.51}$$

We can do similar calculations for the second and fourth terms and it is easy to check that they produce the imaginary parts of box integrals.

The cut C_{12}

Here we consider the cut C_{12}. The integrand consists of the three terms. Integration of the first two terms can be performed by the same procedure as presented earlier in the paper. Therefore, we will only state the results. The first two terms contribute to box coefficients, which can be neglected for our purposes, and to the bubble coefficient $c_{2:2;1}$. The corresponding contributions are

$$c_{2:2;1}^{(1)} = \frac{[4\ 6]^4 \langle 1\ 3 \rangle^2}{[4\ 5][5\ 6]\langle 3|P_{456}|6]P_{456}^2} \frac{\langle 3|P_{456}|4]\langle 4|P_{456}|2]}{\langle 2|P_{456}|4]\langle 4|P_{456}P_{12}P_{456}|4]} \qquad (4.52)$$

and

$$c_{2:2;1}^{(2)} = \frac{\langle 3\ 5 \rangle^4 [2\ 6]^2}{\langle 3\ 4 \rangle \langle 4\ 5 \rangle \langle 3|P_{345}|6]P_{345}^2} \frac{\langle 1|P_{345}|5]\langle 5|P_{345}|6]}{\langle 5|P_{345}|1]\langle 5|P_{345}P_{12}P_{345}|5]}. \qquad (4.53)$$

Now we consider the last term. After performing the t-integration, its contribution to the cut C_{12} can be written as follows

$$C_{12}^{(3)} = C \int \frac{\langle \ell\ d\ell \rangle [\ell\ d\ell]}{\langle \ell|P_{12}|\ell]^2} P_{12}^2 g(\ell) \frac{[\ell\ 2]}{[\ell\ 1]} \left(-\frac{\langle \ell|P_{12}|\ell\rangle\langle 5|6|4]}{P_{12}^2\langle \ell|Q|\ell]} + \frac{\langle 5\ \ell\rangle[\ell\ 4]}{\langle \ell|Q|\ell]} \right), \qquad (4.54)$$

where

$$C = \frac{1}{\langle 1\ 2 \rangle[3\ 4]\langle 5\ 6 \rangle}, \qquad (4.55)$$

$$g(\ell) = -\frac{\langle \ell\ 5 \rangle}{\langle \ell\ 6 \rangle} \frac{\langle \ell|P_{12}|4]}{\langle \ell|P_{12}P_{34}|5\rangle} \frac{(\langle 5|6|4]\langle 1\ \ell \rangle + \langle 1|2|4]\langle 5\ \ell \rangle)^2}{\langle \ell|P_{56}|4]\langle \ell|P_{12}|3]} \qquad (4.56)$$

and

$$Q = \frac{1}{P_{12}^2}(P_{56}^2 P_{12} + P_{12}^2 P_{56}). \qquad (4.57)$$

Now we use our method to split the integrand in (4.54). We find that

$$C_{12}^{(3)} = C \int \langle \ell\ d\ell \rangle [\ell\ d\ell] \left[\frac{1}{\langle \ell|P_{12}|\ell][1\ \ell]} \frac{g(\ell)[1\ 2]}{\langle \ell|Q|1]} \left(\langle 5|6|4] - \frac{\langle 1\ 2 \rangle[4\ 1]\langle 5\ \ell \rangle}{\langle \ell\ 2 \rangle} \right) \right.$$
$$- \frac{1}{\langle \ell|P_{12}|\ell]^2} \frac{g(\ell)P_{12}^2\langle 5\ \ell \rangle\langle \ell|P_{12}|4]\langle \ell\ 1 \rangle}{\langle \ell\ 2 \rangle\langle \ell|P_{12}Q|\ell\rangle} \qquad (4.58)$$
$$\left. - \frac{1}{\langle \ell|P_{12}|\ell]\langle \ell|Q|\ell\rangle} \frac{g(\ell)\langle \ell|Q|2]}{\langle \ell|Q|1]\langle \ell|P_{12}Q|\ell\rangle} (\langle 5|6|4]\langle \ell|P_{12}Q|\ell\rangle + P_{12}^2\langle 5\ell\rangle\langle \ell|Q|4]) \right].$$

29

Among these three parts, the first one contributes to one of the box coefficients and can be neglected for our purposes. The second term gives a contribution to the bubble coefficient $c_{2:2;1}$. Performing the ℓ-integration, we find this contribution to be of the following form

$$c_{2:2;1}^{(3)} = -\frac{1}{[3\,4]\langle 5\,6\rangle} \sum_{i=1}^{6} \left(\langle \ell\,\ell_i\rangle \frac{g(\ell)\langle 5\,\ell\rangle\langle \ell|P_{12}|4][2\,\ell]}{\langle \ell\,2\rangle\langle \ell|P_{12}Q|\ell\rangle\langle \ell|P_{12}|\ell]} \right)_{\ell \to \ell_i}. \qquad (4.59)$$

Here the locations of the six poles is as follows:

$$|\ell_1\rangle = |P_{56}|4], \quad |\ell_2\rangle = |P_{12}|3], \quad |\ell_3\rangle = |6\rangle, \quad |\ell_4\rangle = |P_{12}P_{34}|5\rangle,$$
$$|\ell_5\rangle = |a\rangle + x_-|b\rangle, \quad |\ell_6\rangle = |a\rangle + x_+|b\rangle, \qquad (4.60)$$

with

$$x_\pm = \frac{-(\langle a|P_{12}Q|b\rangle + \langle b|P_{12}Q|a\rangle) \pm \langle a\,b\rangle\sqrt{\Delta_{3m}}}{2\langle b|P_{12}Q|b\rangle} \qquad (4.61)$$

and

$$\Delta_{3m} = \left((P_{12}^2)^2 + (P_{34}^2)^2 + (P_{56}^2)^2 - 2P_{12}^2 P_{34}^2 - 2P_{34}^2 P_{56}^2 - 2P_{56}^2 P_{12}^2 \right) \qquad (4.62)$$

Note that the pole $|\ell\rangle = |2\rangle$ does not contribute because of the factor $[2\,\ell]$ in the numerator of (4.59).

Now we move on to the last term in (4.58). It will produce the contribution to the three-mass triangle coefficient. By introducing the Feynman parameter z, we can write it as follows:

$$C_{12}^{(3,3)} = C \int_0^1 dz \int \frac{\langle \ell d\,\ell\rangle[\ell\,d\ell]}{\langle \ell|P^2|\ell]} \frac{-g(\ell)\langle \ell|Q|2]}{\langle \ell|Q|1]\langle \ell|P_{12}Q|\ell\rangle} (\langle 5|6|4]\langle \ell|P_{12}Q|\ell\rangle + P_{12}^2\langle 5\,\ell\rangle\langle \ell|Q|4]), \qquad (4.63)$$

where

$$P = (1-z)P_{12} + zQ. \qquad (4.64)$$

Performing the ℓ-integration, we obtain

$$C_{12}^{(3,3)} = -C \int_0^1 dz \left[\frac{[\eta\,\ell]}{\langle \ell|P|\ell]\langle \ell|P|\eta]} \frac{-g(\ell)\langle \ell|Q|2]}{\langle \ell|Q|1]\langle \ell|P_{12}Q|\ell\rangle} (\langle 5|6|4]\langle \ell|P_{12}Q|\ell\rangle + P_{12}^2\langle 5\,\ell\rangle\langle \ell|Q|4]) \right]_{\text{pole}}, \qquad (4.65)$$

where $|\eta]$ is an arbitrary auxiliary spinor. Let us write $|\eta]$ as

$$|\eta] = |P|\widetilde{\eta}\rangle \qquad (4.66)$$

for some spinor $|\widetilde{\eta}\rangle$. Then (4.65) becomes

$$C_{12}^{(3,3)} = C \int_0^1 dz \left[\frac{\langle\widetilde{\eta}|P|\ell]}{\langle\ell|P|\ell]P^2} \frac{g(\ell)\langle\ell|Q|2]}{\langle\ell\ \widetilde{\eta}\rangle\langle\ell|Q|1]\langle\ell|P_{12}Q|\ell\rangle} (\langle5|6|4]\langle\ell|P_{12}Q|\ell\rangle + P_{12}^2\langle5\ \ell\rangle\langle\ell|Q|4]) \right]_{\text{pole}}.$$

$$(4.67)$$

The Feynman integral of this type was considered before. It produces the functions $R_1(a_j, b_j, c_j)$ and $R_2(a_j, b_j, c_j)$. As a result, the contribution to the three-mass triangle coefficient is given by

$$c_{3:2:2;1}^{3m} = \frac{1}{\langle1\ 2\rangle[3\ 4]\langle5\ 6\rangle} \sum_{i=1}^6 \left[\langle\ell\ell_i\rangle \frac{g(\ell)\langle\ell|Q|2]R_1(a_j,b_j,c_j)}{\langle\ell\ \widetilde{\eta}\rangle\langle\ell|Q|1]\langle\ell|P_{12}Q|\ell\rangle} (\langle5|6|4]\langle\ell|P_{12}Q|\ell\rangle + P_{12}^2\langle5\ \ell\rangle\langle\ell|Q|4]) \right]_{\ell\to\ell_i}$$

$$+ \frac{1}{\langle1\ 2\rangle[3\ 4]\langle5\ 6\rangle} \sum_{i=7}^8 \left[\langle\ell\ell_i\rangle \frac{g(\ell)\langle\ell|Q|2]R_2(a_j,b_j,c_j)}{\langle\ell\ \widetilde{\eta}\rangle\langle\ell|Q|1]\langle\ell|P_{12}Q|\ell\rangle} (\langle5|6|4]\langle\ell|P_{12}Q|\ell\rangle + P_{12}^2\langle5\ \ell\rangle\langle\ell|Q|4]) \right]_{\ell\to\ell_i},$$

$$(4.68)$$

where, in this example, the coefficients a_j, b_j and c_j are

$$a_0 = (Q - P_{12})^2, \quad a_1 = 2P_{12} \cdot (Q - P_{12}), \quad a_2 = P_{12}^2,$$

$$b_1 = \langle\ell|Q - P_{12}|\ell], \quad b_2 = \langle\ell|P_{12}|\ell], \quad c_1 = \langle\widetilde{\eta}|(Q - P_{12})|\ell], \quad c_2 = \langle\widetilde{\eta}|P_{12}|\ell].$$

$$(4.69)$$

In the first term in eq. (4.68), the summation is over the poles

$$|\ell_1\rangle = |P_{56}|4], \quad |\ell_2\rangle = |P_{12}|3], \quad |\ell_3\rangle = |6\rangle, \quad |\ell_4\rangle = |P_{12}P_{34}|5\rangle,$$

$$|\ell_5\rangle = |\widetilde{\eta}\rangle, \quad |\ell_6\rangle = |Q|1].$$

$$(4.70)$$

In the second term, the summation is over the poles

$$|\ell_7\rangle = |a\rangle + x_-|b\rangle, \quad |\ell_8\rangle = |a\rangle + x_+|b\rangle. \qquad (4.71)$$

Thus, the bubble coefficient $c_{2:2;1}$ is represented by the sum

$$c_{2:2;1} = c_{2:2;1}^{(1)} + c_{2:2;1}^{(2)} + c_{2:2;1}^{(3)}, \qquad (4.72)$$

where $c_{2:2;1}^{(1)}$, $c_{2:2;1}^{(2)}$ and $c_{2:2;1}^{(3)}$ are given by (4.52), (4.53) and (4.59) respectively. The three-mass triangle coefficient $c_{3:2:2;1}^{3m}$ is given by (4.68).

4.4. The amplitude $A(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$

The next-to-MHV n-point amplitude with all negative-helicity gluons appearing consecutively has been computed in [23]. We perform the calculation here as well, as an illustration of our procedure. The solution will emerge in a different form.

From the viewpoint of our discussion in Section 2, there are no box and three-mass triangle contributions, hence no one-mass and two-mass triangle contributions. All we have are bubble contributions which can be calculated by double cuts. There are two nonvanishing double cuts, which we denote by I and II.

$$
\begin{aligned}
\text{I} &= \int d\mu \; A_L(\ell_1, k+1, \ldots, n, 1, 2, \ell_2) A_R(\ell_2, 3, 4, \ldots, k, \ell_1) \quad 4 \le k \le n-1 \\
\text{II} &= \int d\mu \; A_L(\ell_1, 2, 3, \ldots, \widetilde{k}, \ell_2) A_R(\ell_2, \widetilde{k}+1, \ldots, 1, \ell_1) \quad 4 \le \widetilde{k} \le n-1
\end{aligned}
\tag{4.73}
$$

However, these two kinds of cuts are mapped to each other under the permutation P_α : $(n-i+4) \leftrightarrow i$ and the identification $\widetilde{k} = n-k+3$.

Now we focus on the case I. To do this we need to know the tree-level amplitudes with two fermions or complex scalars, which we list in Appendix B. First we have

$$
\begin{aligned}
&A_L(\ell_1^+, (k+1)^+, \ldots, n^+, 1^-, 2^-, \ell_2^-) \\
&= -\sum_{j=0}^{n-k-2} \frac{\langle n-1-j \; n-j\rangle \langle 1|K_{n-j}^{[j+2]} K_{n-j}^{[j+3]}|\ell_2\rangle^3}{\langle \ell_2 \, \ell_1\rangle \langle \ell_1 \, k+1\rangle \langle k+1 \, k+2\rangle \cdots \langle n \, 1\rangle t_{n-j}^{[j+2]} t_{n-j}^{[j+3]} \langle n-j|K_{n-j}^{[j+2]}|2\rangle \langle n-1-j|K_{n-j}^{[j+2]}|2\rangle} \\
&\quad \times \left(-\frac{\langle 1|K_{n-j}^{[j+2]} K_{n-j}^{[j+3]}|\ell_1\rangle}{\langle 1|K_{n-j}^{[j+2]} K_{n-j}^{[j+3]}|\ell_2\rangle} \right)^a \\
&\quad - \frac{\langle \ell_1 \, k+1\rangle \langle 1|K_{k+1}^{[n+2-(k+1)]} K_{k+1}^{[n+3-(k+1)]}|\ell_2\rangle^3}{\langle \ell_2 \, \ell_1\rangle \langle \ell_1 \, k+1\rangle \langle k+1 \, k+2\rangle \ldots \langle n|1\rangle t_{k+1}^{[n+2-(k+1)]} t_{k+1}^{[n+3-(k+1)]} \langle k+1|K_{k+1}^{[n+2-(k+1)]}|2\rangle} \\
&\quad \times \frac{1}{\langle \ell_1|K_{k+1}^{[n+2-(k+1)]}|2\rangle} \left(-\frac{\langle 1|K_{k+1}^{[n+2-(k+1)]} K_{k+1}^{[n+3-(k+1)]}|\ell_1\rangle}{\langle 1|K_{k+1}^{[n+2-(k+1)]} K_{k+1}^{[n+3-(k+1)]}|\ell_2\rangle} \right)^a
\end{aligned}
\tag{4.74}
$$

where the last term is the special case with $j = n-k-1$. In this formula, $a = 0, 1, 2$ for ℓ_1, ℓ_2 to be gluons, fermions and complex scalars. Using (4.74) and MHV-amplitude of A_R we get the integrand of cut I as

$$
\begin{aligned}
\text{I} &= -\int d\mu \frac{\langle k \, k+1\rangle \langle 1 \, 2\rangle \langle 2 \, 3\rangle}{\prod_{i=1}^{n} \langle i \, i+1\rangle} \frac{\langle \ell_1 \, 3\rangle}{\langle \ell_1 \, k+1\rangle \langle k \, \ell_1\rangle} \\
&\quad \times \sum_{j=0}^{n-k-1} \frac{\langle n-1-j \; n-j\rangle \langle 1|K_{n-j}^{[j+2]} K_{n-j}^{[j+3]}|3\rangle^2 \langle 1|K_{n-j}^{[j+2]} K_{n-j}^{[j+3]}|\ell_1\rangle}{t_{n-j}^{[j+2]} t_{n-j}^{[j+3]} \langle n-j|K_{n-j}^{[j+2]}|2\rangle \langle n-1-j|K_{n-j}^{[j+2]}|2\rangle}
\end{aligned}
\tag{4.75}
$$

Notice that in the above integration, only the holomorphic part $|\ell_1\rangle$ appears in the integrand, so the integration is very easy to do, similar to the example $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$. We separate the integral into the cases where $j \neq n - k - 1$ and the special case of $j = n - k - 1$. For the cases where $j \neq n - k - 1$ there are three potential poles: $|\ell_1\rangle = K_3^{[k-2]}|\eta], |\ell_1\rangle = |k+1\rangle$ and $|\ell_1\rangle = |k\rangle$. By choosing $|\eta] = |K_3^{[k-2]}|3\rangle$, we get rid of one of them. Similarly, for the case $j = n - k - 1$, we take $|\eta] = K_3^{[k-2]}|3\rangle$ to get rid of one pole and leave only two poles: $|\ell_1\rangle = |k\rangle$ and $|\ell_1\rangle = |K_2^{[k-1]}|2]$. Adding all these pieces together, we find that the coefficient from cut I is

$$
\begin{aligned}
c_{2:k-2;3} = & \frac{\langle 1\,2\rangle\langle 2\,3\rangle}{\prod_{i=1}^{n}\langle i\,i+1\rangle} \sum_{j=0}^{n-k-2} \frac{\langle n-1-j\,\,n-j\rangle\langle 1|K_{n-j}^{[j+2]}K_{n-j}^{[j+3]}|3\rangle^2}{t_{n-j}^{[j+2]}t_{n-j}^{[j+3]}\langle n-j|K_{n-j}^{[j+2]}|2]\langle n-1-j|K_{n-j}^{[j+2]}|2]} \\
& \times \left(\frac{\langle 3|K_3^{[k-2]}|k+1\rangle\langle k+1|K_{n-j}^{[j+3]}K_{n-j}^{[j+2]}|1\rangle}{\langle k+1|K_3^{[k-2]}|k+1]} - \frac{\langle 3|K_3^{[k-2]}|k\rangle\langle k|K_{n-j}^{[j+3]}K_{n-j}^{[j+2]}|1\rangle}{\langle k|K_3^{[k-2]}|k]} \right) \\
& + \frac{\langle 1\,2\rangle\langle 2\,3\rangle}{\prod_{i=1}^{n}\langle i\,i+1\rangle} \frac{\langle k\,k+1\rangle\langle 1|K_2^{[k-1]}K_3^{[k-2]}|3\rangle^2}{t_2^{[k-1]}t_3^{[k-2]}\langle k+1|K_2^{[k-1]}|2]\langle k|K_2^{[k-1]}|2]} \\
& \times \left(-\frac{\langle 3|K_3^{[k-2]}|k\rangle\langle k|K_3^{[k-2]}K_2^{[k-1]}|1\rangle}{\langle k|K_3^{[k-2]}|k]} - \frac{\langle 2|K_2^{[k-1]}K_3^{[k-2]}|3\rangle\langle 1|K_3^{[k-2]}|2]}{\langle 2|K_3^{[k-2]}|2]} \right).
\end{aligned}
\tag{4.76}
$$

By symmetry we read out the coefficient from cut II as

$$
\begin{aligned}
c_{2:k-1;2} = & -\frac{\langle 1\,2\rangle\langle 2\,3\rangle}{\prod_{i=1}^{n}\langle i\,i+1\rangle} \sum_{j=0}^{n-k-2} \frac{\langle j+5\,\,j+4\rangle\langle 3|K_3^{[j+2]}K_2^{[j+3]}|1\rangle^2}{t_3^{[j+2]}t_2^{[j+3]}\langle j+4|K_3^{[j+2]}|2]\langle j+5|K_3^{[j+2]}|2]} \\
& \times \left(\frac{\langle 1|K_{n-k+4}^{[k-2]}|n-k+3\rangle\langle n-k+3|K_2^{[j+3]}K_3^{[j+2]}|3\rangle}{\langle n-k+3|K_{n-k+4}^{[k-2]}|n-k+3]} \right. \\
& \left. -\frac{\langle 1|K_{n-k+4}^{[k-2]}|n-k+4\rangle\langle n-k+4|K_2^{[j+3]}K_3^{[j+2]}|3\rangle}{\langle n-k+4|K_{n-k+4}^{[k-2]}|n-k+4]} \right) \\
& + \frac{\langle 1\,2\rangle\langle 2\,3\rangle}{\prod_{i=1}^{n}\langle i\,i+1\rangle} \frac{\langle n-k+4\,\,n-k+3\rangle\langle 3|K_{n-k+4}^{[k-1]}K_{n-k+4}^{[k-2]}|1\rangle^2}{t_{n-k+4}^{[k-1]}t_{n-k+4}^{[k-2]}\langle n-k+3|K_{n-k+4}^{[k-1]}|2]\langle n-k+4|K_{n-k+4}^{[k-1]}|2]} \\
& \times \left(\frac{\langle 1|K_{n-k+4}^{[k-2]}|n-k+4\rangle\langle n-k+4|K_{n-k+4}^{[k-2]}K_{n-k+4}^{[k-1]}|3\rangle}{\langle n-k+4|K_{n-k+4}^{[k-2]}|k-k+4]} \right. \\
& \left. + \frac{\langle 2|K_{n-k+4}^{[k-1]}K_{n-k+4}^{[k-2]}|1\rangle\langle 3|K_{n-k+4}^{[k-2]}|2]}{\langle 2|K_{n-k+4}^{[k-2]}|2]} \right).
\end{aligned}
\tag{4.77}
$$

In [23], the amplitude was decomposed in terms of functions K_0 and L_0, given here in (4.13), where the function K_0 is proportional to the bubble integral and the function L_0 is related to the Feynman parameter integral for a two-mass triangle integral. By the identity (4.14), it is possible to convert their expression to an expansion in bubble integrals only, as we have done here.

5. Summary Of Results For Next-To-MHV Six-Gluon Amplitudes

In this section we collect our results for the next-to-MHV six-gluon amplitudes for the reader's convenience.

We define the following functions:

$$
\begin{aligned}
R_1(a_j, b_j, c_j) &= \frac{(2a_2b_1c_1 - a_1b_2c_1 - a_1b_1c_2 + 2a_0b_2c_2)}{2(a_2b_1^2 - a_1b_1b_2 + a_0b_2^2)} \\
R_2(a_j, b_j, c_j) &= \frac{(a_2b_2c_1 + a_2b_1c_2 - a_1b_2c_2)}{b_2(2a_2b_1 - a_1b_2)}
\end{aligned}
\tag{5.1}
$$

5.1. $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

The amplitude is given by

$$
A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = \frac{r_\Gamma(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(c_{2:3;6}I_{2:3;6} + c_{2:3;2}I_{2:3;2} + c_{2:2;3}I_{2:2;3} + c_{2:2;6}I_{2:2;6} \right)
\tag{5.2}
$$

with $r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$ and

$$
\begin{aligned}
c_{2:3;6} &= -\frac{\langle 3|P_{612}|6]^2}{[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 5|P_{612}|2]P_{612}^2} \left(\frac{\langle 3|P_5|6]P_{612}^2}{\langle 5|P_{612}|5]} + \frac{\langle 3|P_{612}P_2P_{612}|6]}{\langle 2|P_{612}|2]} \right) \\
c_{2:3;2} &= -\frac{\langle 1|P_{561}|4]^2}{\langle 5\,6\rangle\langle 6\,1\rangle[2\,3][3\,4]\langle 5|P_{561}|2]P_{561}^2} \left(\frac{\langle 1|P_2|4]P_{561}^2}{\langle 2|P_{561}|2]} + \frac{\langle 1|P_{561}P_5P_{561}|4]}{\langle 5|P_{561}|5]} \right) \\
c_{2:2;3} &= \frac{[4\,5]\langle 5|P_{345}|6]\langle 3|P_{345}|6]^2}{\langle 4\,5\rangle[6\,1][1\,2]\langle 5|P_{345}|5]\langle 5|P_{345}|2]P_{345}^2} + \frac{\langle 2\,3\rangle\langle 1|P_{234}|2]\langle 1|P_{234}|4]^2}{[2\,3]\langle 5\,6\rangle\langle 6\,1\rangle\langle 2|P_{234}|2]\langle 5|P_{234}|2]P_{234}^2} \\
c_{2:2;6} &= \frac{[5\,6]\langle 5|P_{561}|4]\langle 1|P_{561}|4]^2}{\langle 5\,6\rangle[2\,3][3\,4]\langle 5|P_{561}|2]\langle 5|P_{561}|5]P_{561}^2} + \frac{\langle 1\,2\rangle\langle 3|P_{612}|2]\langle 3|P_{612}|6]^2}{[1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 5|P_{612}|5]\langle 2|P_{612}|2]P_{612}^2}
\end{aligned}
\tag{5.3}
$$

5.2. $A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$

The amplitude is given by

$$A(1^-, 2^-, 3^+, 4^-, 5^+, 6^+) = \frac{r_\Gamma(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(c^{2m\ h}_{4:2;2} I^{2m\ h}_{4F:2;2} + c^{2m\ h}_{4:2;4} I^{2m\ h}_{4F:2;4} + c^{2m\ h}_{4:2;6} I^{2m\ h}_{4F:2;6} + c^{1m}_{4;5} I^{1m}_{4F;5} + c^{1m}_{4;6} I^{1m}_{4F;6} \right.$$

$$+ c^{3m}_{3:2:2;2} I^{3m}_{3:2:2;2} + c_{2:3;2} I_{2:3;2} + c_{2:3;6} I_{2:3;6} + c_{2:3;1} I_{2:3;1}$$

$$\left. + c_{2:2;2} I_{2:2;2} + c_{2:2;3} I_{2:2;3} + c_{2:2;4} I_{2:2;4} + c_{2:2;6} I_{2:2;6} \right)$$

$$\tag{5.4}$$

with

$$c^{2m\ h}_{4:2;2} = - \frac{P^2_{61} P^2_{456}}{2} \frac{\langle 4|P_{456}|3]^2 \langle 4\ 6\rangle [3\ 1]}{\langle 6|P_{456}|1]^2 \langle 4\ 5\rangle \langle 5\ 6\rangle [1\ 2][2\ 3]}$$

$$c^{2m\ h}_{4:2;4} = - \frac{P^2_{23} P^2_{612}}{2} \frac{[2\ 6]\langle 4|P_{612}|2]\langle 4|P_{612}|6]^2}{\langle 4\ 5\rangle [6\ 1][1\ 2]\langle 5|P_{612}|2]\langle 3|P_{612}|2]^2}$$

$$c^{2m\ h}_{4:2;6} = - \frac{P^2_{45} P^2_{561}}{2} \frac{\langle 1\ 5\rangle\langle 1|P_{561}|3]^2\langle 5|P_{561}|3]}{[2\ 3]\langle 5\ 6\rangle\langle 6\ 1\rangle\langle 5|P_{561}|2]\langle 5|P_{561}|4]^2} \tag{5.5}$$

$$c^{1m}_{4;5} = - \frac{P^2_{23} P^2_{34}}{2} \frac{\langle 1|P_{234}|2]\langle 1|P_{234}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle [4\ 2]^2\langle 5|P_{234}|2] P^2_{234}}$$

$$c^{1m}_{4;6} = - \frac{P^2_{34} P^2_{45}}{2} \frac{\langle 5|P_{345}|6]\langle 4|P_{345}|6]^2}{[6\ 1][1\ 2]\langle 3\ 5\rangle^2\langle 5|P_{345}|2] P^2_{345}}$$

$$c^{3m}_{3:2:2;2} = \frac{1}{[4\ 5]\langle 6\ 1\rangle\langle 2\ 3\rangle} \times$$

$$\sum_{i=1,2,3,6,7} \left[\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle \ell|Q|3] R_1(a_j, b_j, c_j)}{\langle \ell\ \widetilde\eta\rangle\langle \ell|Q|2]\langle \ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle \ell|P_{23}Q|\ell\rangle + P^2_{23}\langle 1\ \ell\rangle\langle \ell|Q|5] \right) \right]_{\ell \to \ell_i}$$

$$+ \frac{1}{[4\ 5]\langle 6\ 1\rangle\langle 2\ 3\rangle} \times$$

$$\sum_{i=4,5} \left[\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle \ell|Q|3] R_2(a_j, b_j, c_j)}{\langle \ell\ \widetilde\eta\rangle\langle \ell|Q|2]\langle \ell|P_{23}Q|\ell\rangle} \left(\langle 1|6|5]\langle \ell|P_{23}Q|\ell\rangle + P^2_{23}\langle 1\ \ell\rangle\langle \ell|Q|5] \right) \right]_{\ell \to \ell_i} .$$

$$\tag{5.6}$$

$$c_{2:3;2} = -\frac{\langle 1|P_{234}|3]^2}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2} \left(\frac{\langle 1|6|5]\langle 5|P_{561}|3]^2}{\langle 5|P_{561}|2]\langle 5|P_{561}|4]\langle 5|P_{561}|5]} \right.$$

$$\left. + \frac{\langle 1\ 2\rangle[2\ 3]^2 P_{561}^2}{[2\ 4]\langle 5|P_{561}|2]\langle 2|P_{561}|2]} + \frac{\langle 1\ 4\rangle[3\ 4]^2 P_{561}^2}{[4\ 2]\langle 5|P_{561}|4]\langle 4|P_{561}|4]} \right)$$

$$c_{2:3;6} = -\frac{\langle 4|P_{612}|6]^2}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P_{612}^2} \left(\frac{\langle 2|1|6]\langle 4|P_{612}|2]^2}{\langle 5|P_{612}|2]\langle 3|P_{612}|2]\langle 2|P_{612}|2]} \right.$$

$$\left. + \frac{[5\ 6]\langle 4\ 5\rangle^2 P_{612}^2}{\langle 3\ 5\rangle\langle 5|P_{612}|2]\langle 5|P_{612}|5]} + \frac{[6\ 3]\langle 3\ 4\rangle^2 P_{612}^2}{\langle 3\ 5\rangle\langle 3|P_{612}|2]\langle 3|P_{612}|3]} \right) \qquad (5.7)$$

$$c_{2:3;1} = \frac{\langle 4|P_{456}|3]^2}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle} \left[\frac{[6\ 3]\langle 6\ 4\rangle}{\langle 6|P_{123}|6]\langle 6|P_{123}|1]} - \frac{\langle 1|2|3]\langle 4|P_{123}|1]}{\langle 1|P_{123}|1]\langle 6|P_{123}|1] P_{123}^2} \right]$$

$$c_{2:2;2} = \frac{\langle 2\ 4\rangle^2[5\ 6]^4}{[5\ 6][6\ 1]P_{561}^2} \frac{\langle 4|P_{561}|5]}{\langle 4|P_{561}|1]} \frac{\langle 5|P_{561}|3]}{\langle 3|P_{561}|5]} \frac{1}{\langle 5|P_{561}P_{23}P_{561}|5]}$$

$$+ \frac{\langle 4|P_{456}|3]^2}{[2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{456}^2\langle 4|P_{456}|1]} \left(-\frac{[3\ 2]\langle 2\ 1\rangle\langle 4|P_{456}|1]^2}{[2\ 1]\langle 6|P_{456}|1]\langle 1|P_{23}|1]} \right.$$

$$\left. + \frac{\langle 4\ 6\rangle^2[2\ 3]\langle 2|P_{456}|6](P_{456}^2)^2}{\langle 6|P_{456}|2]\langle 6|P_{456}|1]\langle 6|P_{456}P_{23}P_{456}|6]} \right)$$

$$- \frac{1}{[4\ 5]\langle 6\ 1\rangle} \sum_{i=1}^{5} \lim_{\ell \to \ell_i} \left[\langle \ell\ \ell_i\rangle \frac{g(\ell)\langle 1\ \ell\rangle\langle \ell|P_{23}|5][3\ \ell]}{\langle \ell\ 3\rangle\langle \ell|P_{23}Q|\ell\rangle\langle \ell|P_{23}|\ell]} \right]$$

$$c_{2:2;3} = \frac{[3\ 5]\langle 5|P_{612}|6]\langle 4|P_{612}|6]^2}{[6\ 1][1\ 2]\langle 3\ 5\rangle\langle 5|P_{612}|2]\langle 5|P_{612}|5] P_{612}^2} + \frac{\langle 2\ 4\rangle\langle 1|P_{561}|2]\langle 1|P_{561}|3]^2}{\langle 5\ 6\rangle\langle 6\ 1\rangle[2\ 4]\langle 5|P_{561}|2]\langle 2|P_{561}|2] P_{561}^2}$$

$$c_{2:2;4} = P_{\alpha}(c_{2:2;2})$$

$$c_{2:2;6} = A_{\text{tree}} - c_{2:2;2} - c_{2:2;3} - c_{2:2;4} - c_{2:3;2} - c_{2:3;6} - c_{2:3;1}$$

$$(5.8)$$

The symmetric action is $P_{\alpha} : i \leftrightarrow 7 - i$ plus conjugation, i.e, $\langle\ \rangle \leftrightarrow [\ ]$. For (5.6) and (5.8), we have defined

$$a_0 = (Q - P_{23})^2, \qquad a_1 = 2P_{23} \cdot (Q - P_{23}), \qquad a_2 = P_{23}^2$$

$$b_1 = \langle \ell|(Q - P_{23})|\ell], \qquad b_2 = \langle \ell|P_{23}|\ell] \qquad c_1 = \langle \widetilde{\eta}|(Q - P_{23})|\ell], \qquad c_2 = \langle \widetilde{\eta}|P_{23}|\ell] \qquad (5.9)$$

$$|\ell_1\rangle = |P_{23}|4], \quad |\ell_2\rangle = |P_{61}|5], \qquad |\ell_3\rangle = |P_{23}P_{45}|6\rangle$$

$$|\ell_4\rangle = |a\rangle + x_+|b\rangle, \qquad |\ell_5\rangle = |a\rangle + x_-|b\rangle, \qquad |\ell_6\rangle = |Q|2], \qquad |\ell_7\rangle = |\widetilde{\eta}\rangle \qquad (5.10)$$

$$x_{\pm} = \frac{-(\langle a|P_{23}Q|b\rangle + \langle b|P_{23}Q|a\rangle) \pm \langle a\ b\rangle\sqrt{\Delta_{3m}}}{2\langle b|P_{23}Q|b\rangle} \qquad (5.11)$$

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where

$$\Delta_{3m} = (P_{61}^2)^2 + (P_{45}^2)^2 + (P_{23}^2)^2 - 2P_{61}^2 P_{23}^2 - 2P_{23}^2 P_{45}^2 - 2P_{45}^2 P_{61}^2$$

$$g(\ell) = -\frac{\langle\ell|P_{23}|5]}{\langle\ell|P_{23}|4]} \frac{(\langle 1|6|5]\langle 2\,\ell\rangle + \langle 2|3|5]\langle 1\,\ell\rangle)^2}{\langle\ell|P_{61}|5]\langle\ell|P_{23}P_{45}|6\rangle}$$

$$Q = \frac{1}{P_{23}^2}(P_{61}^2 P_{23} + P_{23}^2 P_{61})$$

(5.12)

Here $|a\rangle$, $|b\rangle$ and $|\widetilde{\eta}\rangle$ are arbitrary.

5.3. $A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$

This helicity configuration has the largest symmetry, namely a $\mathbb{Z}_6$ generated by P_α :

$i \to i + 1$ plus conjugation, so we have grouped everything into orbits. The amplitude is

given by

$$A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+) = \frac{r_\Gamma(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(\sum_{i=1}^6 c_{4:2;i}^{2m\ h} I_{4F:2;i}^{2m\ h} + \sum_{i=1}^6 c_{4;i}^{1m} I_{4F;i}^{1m} + \sum_{i=1}^2 c_{3:2:2;i}^{3m} I_{3:2:2;i}^{3m} \right.$$

$$\left. + \sum_{i=1}^3 c_{2:3;i} I_{2:3;i} + \sum_{i=1}^6 c_{2:2;i} I_{2:2;i} \right)$$

(5.13)

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with

$$c_{4:2;1}^{2m\;h} = \frac{P_{345}^2 P_{56}^2}{2} \frac{\langle 1|P_{345}|4]^2}{\langle 1\,2\rangle[3\,4]\langle 6|P_{345}|5]^2} \frac{\langle 1|P_{345}|5]\langle 6|P_{345}|4]}{\langle 2|P_{345}|5]\langle 6|P_{345}|3]}$$

$$c_{4;1}^{1m} = -\frac{P_{45}^2 P_{56}^2}{2} \frac{\langle 5|P_{456}|2]^2}{[1\,2][2\,3]\langle 4\,6\rangle^2 P_{456}^2} \frac{\langle 4|P_{456}|2]\langle 6|P_{456}|2]}{\langle 4|P_{456}|1]\langle 6|P_{456}|3]}$$

$$c_{3:2:2;1}^{3m} = \frac{1}{\langle 1\,2\rangle[3\,4]\langle 5\,6\rangle}$$

$$\times \sum_{i=1,2,3,4,7,8} \left[\langle \ell\ell_i\rangle \frac{g(\ell)\langle\ell|Q|2]R_1(a_j,b_j,c_j)}{\langle\ell\,\widetilde{\eta}\rangle\langle\ell|Q|1]\langle\ell|P_{12}Q|\ell\rangle} (\langle 5|6|4]\langle\ell|P_{12}Q|\ell\rangle + P_{12}^2\langle 5\,\ell\rangle\langle\ell|Q|4]) \right]_{\ell\to\ell_i}$$

$$+ \frac{1}{\langle 1\,2\rangle[3\,4]\langle 5\,6\rangle} \sum_{i=5,6} \left[\langle\ell\ell_i\rangle \frac{g(\ell)\langle\ell|Q|2]R_2(a_j,b_j,c_j)}{\langle\ell\,\widetilde{\eta}\rangle\langle\ell|Q|1]\langle\ell|P_{12}Q|\ell\rangle} (\langle 5|6|4]\langle\ell|P_{12}Q|\ell\rangle + P_{12}^2\langle 5\,\ell\rangle\langle\ell|Q|4]) \right]_{\ell\to\ell_i},$$

$$c_{2:3;1} = -\frac{\langle 5|P_{123}|2]^2}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle P_{123}^2} \left(\frac{\langle 4|P_{123}|2]^2\langle 4\,5\rangle\langle 5\,6\rangle[4\,6]}{\langle 4|P_{123}|1]\langle 4|P_{123}|3]\langle 4|P_{123}|4]\langle 4\,6\rangle} \right.$$

$$- \frac{\langle 1\,5\rangle[1\,2]^2 P_{123}^2\langle 5|P_{123}|1]}{[1\,3]\langle 4|P_{123}|1]\langle 6|P_{123}|1]\langle 1|P_{123}|1]} + \frac{\langle 5|P_{123}|3]}{\langle 6|P_{123}|3]} \frac{\langle 5\,3\rangle[2\,3]^2 P_{123}^2}{[3\,1]\langle 4|P_{123}|3]\langle 3|P_{123}|3]}$$

$$\left. + \frac{\langle 5\,6\rangle}{\langle 6|P_{123}|3]} \frac{\langle 5\,4\rangle[4\,6]\langle 6|P_{123}|2]^2}{\langle 6\,4\rangle\langle 6|P_{123}|1]\langle 6|P_{123}|6]} \right)$$

$$c_{2:2;1} = \frac{[4\,6]^4\langle 1\,3\rangle^2}{[4\,5][5\,6]\langle 3|P_{456}|6]P_{456}^2} \frac{\langle 3|P_{456}|4]\langle 4|P_{456}|2]}{\langle 2|P_{456}|4]\langle 4|P_{456}P_{12}P_{456}|4]}$$

$$+ \frac{\langle 3\,5\rangle^4[2\,6]^2}{\langle 3\,4\rangle\langle 4\,5\rangle\langle 3|P_{345}|6]P_{345}^2} \frac{\langle 1|P_{345}|5]\langle 5|P_{345}|6]}{\langle 5|P_{345}|1]\langle 5|P_{345}P_{12}P_{345}|5]}$$

$$- \frac{1}{[3\,4]\langle 5\,6\rangle} \sum_{i=1}^{6} \lim_{\ell\to\ell_i} \left[\langle\ell\,\ell_i\rangle \frac{g(\ell)\langle 5\,\ell\rangle\langle\ell|P_{12}|4][2\,\ell]}{\langle\ell\,2\rangle\langle\ell|P_{12}Q|\ell\rangle\langle\ell|P_{12}|\ell]} \right]$$

(5.14)

and the other coefficients can be obtained by the symmetric action of P_α. Here one needs

$$|\ell_1\rangle = |P_{56}|4], \quad |\ell_2\rangle = |P_{12}|3], \quad |\ell_3\rangle = |6\rangle, \quad |\ell_4\rangle = |P_{12}P_{34}|5\rangle,$$

$$|\ell_5\rangle = |a\rangle + x_-|b\rangle, \quad |\ell_6\rangle = |a\rangle + x_+|b\rangle, \quad |\ell_7\rangle = |\widetilde{\eta}\rangle, \quad |\ell_8\rangle = |Q|1].$$

(5.15)

$$a_0 = (Q - P_{12})^2, \quad a_1 = 2P_{12}\cdot(Q - P_{12}), \quad a_2 = P_{12}^2,$$

$$b_1 = \langle\ell|Q - P_{12}|\ell], \quad b_2 = \langle\ell|P_{12}|\ell], \quad c_1 = \langle\widetilde{\eta}|(Q - P_{12})|\ell], \quad c_2 = \langle\widetilde{\eta}|P_{12}|\ell]$$

(5.16)

$$g(\ell) = -\frac{\langle\ell\,5\rangle}{\langle\ell\,6\rangle} \frac{\langle\ell|P_{12}|4]}{\langle\ell|P_{12}P_{34}|5\rangle} \frac{(\langle 5|6|4]\langle 1\,\ell\rangle + \langle 1|2|4]\langle 5\,\ell\rangle)^2}{\langle\ell|P_{56}|4]\langle\ell|P_{12}|3]}$$

$$Q = \frac{1}{P_{12}^2}(P_{56}^2 P_{12} + P_{12}^2 P_{56}).$$

(5.17)

$$x_\pm = \frac{-(\langle a|P_{12}Q|b\rangle + \langle b|P_{12}Q|a\rangle) \pm \langle a\ b\rangle \sqrt{\Delta_{3m}}}{2\langle b|P_{12}Q|b\rangle} \tag{5.18}$$

where

$$\Delta_{3m} = (P_{12}^2)^2 + (P_{34}^2)^2 + (P_{56}^2)^2 - 2P_{12}^2 P_{34}^2 - 2P_{34}^2 P_{56}^2 - 2P_{56}^2 P_{12}^2 \tag{5.19}$$

and $|a\rangle, |b\rangle, |\widetilde{\eta}\rangle$ can be chosen arbitrarily.

Acknowledgments

Appendix A. Scalar Integral Functions

In this appendix we list the explicit results for the scalar integrals derived in the reduction procedure of [37]. The expressions here are taken from [3,47].

The dimensional regularization parameter is $\epsilon = (4 - D)/2$. The constant r_Γ is defined by

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \tag{A.1}$$
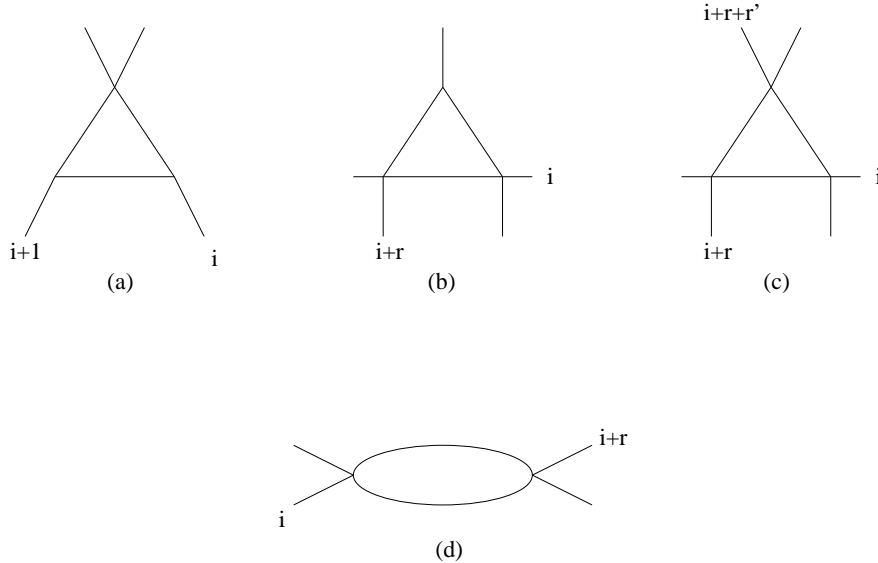

Fig. 3: Scalar bubble and triangle integrals. (a) One-mass triangle $I_{3;i}^{1m}$. (b) Two-mass triangle $I_{3:r;i}^{2m}$. (c) Three-mass triangle $I_{3:r:r';i}^{3m}$. (d) Bubble $I_{2:r;i}$.
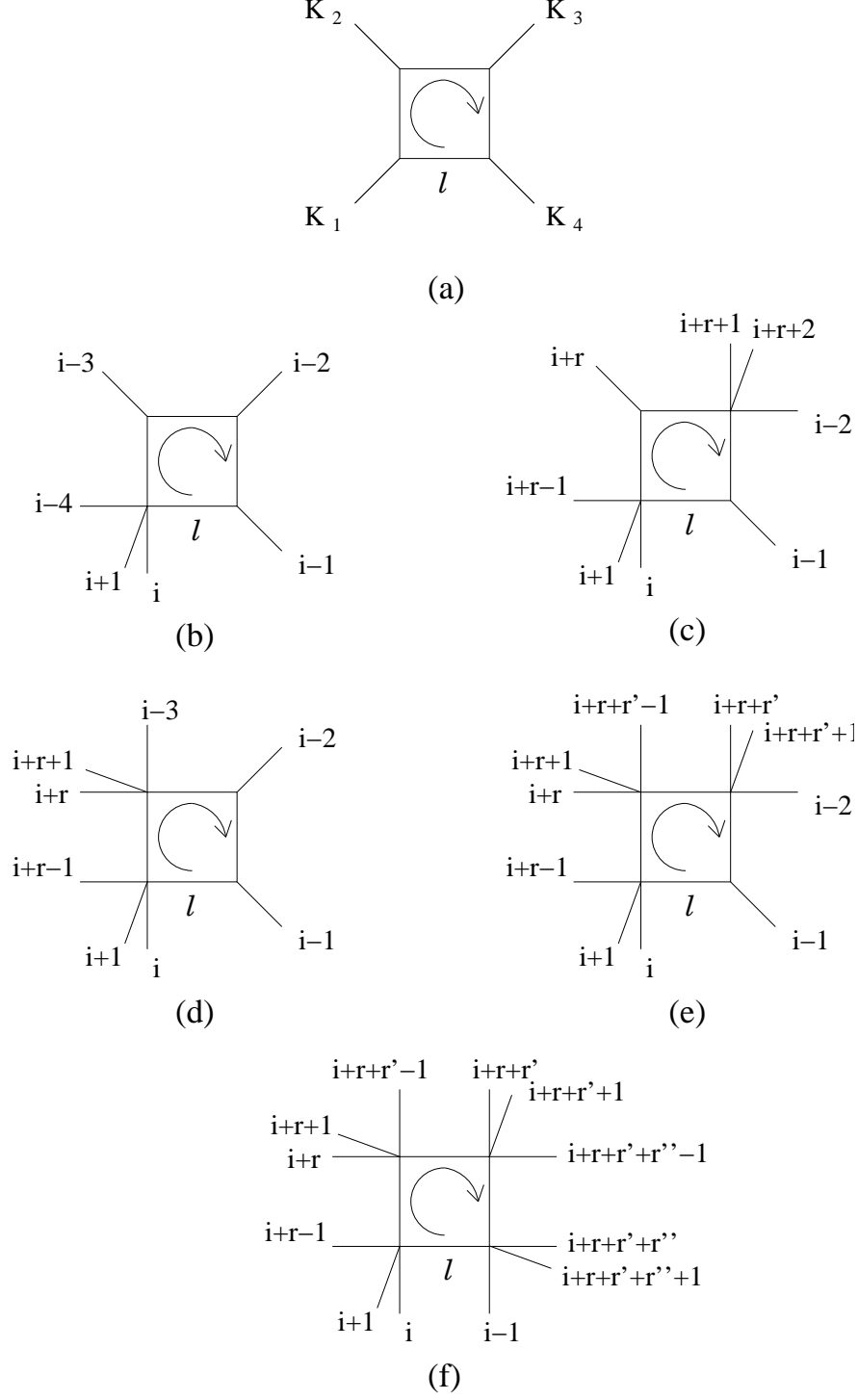
Fig. 4: Scalar box integrals. (a) The outgoing external momenta at each of the vertices are K_1, K_2, K_3, K_4, defined to correspond to sums of the momenta of gluons in the exact orientation shown. (b) One-mass $I_{4;i}^{1m}$. (c) Two-mass "easy" $I_{4:r;i}^{2m\ e}$. (d) Two-mass "hard" $I_{4:r;i}^{2m\ h}$. (e) Three-mass $I_{4:r:r';i}^{3m}$. (f) Four-mass $I_{4:r:r':r";i}^{4m}$.

Scalar bubble integrals:

$$I_2 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-K)^2} \tag{A.2}$$

$$I_{2:r;i} = r_\Gamma \left(\frac{1}{\epsilon} - \ln(-t_i^{[r]}) + 2 \right) + \mathcal{O}(\epsilon) \tag{A.3}$$

Scalar triangle integrals:

$$I_3 = i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-K_1)^2(p+K_3)^2} \tag{A.4}$$

$$
\begin{aligned}
I_{3;i}^{1m} &= \frac{r_\Gamma}{\epsilon^2}(-t_i^{[2]})^{-1-\epsilon} \\
I_{3:r;i}^{2m} &= \frac{r_\Gamma}{\epsilon^2} \frac{(-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[n-r-1]})^{-\epsilon}}{(-t_i^{[r]}) - (-t_{i+r}^{[n-r-1]})} \\
I_{3:r:r';i}^{3m} &= \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^{3} \left[\mathrm{Li}_2\left(-\frac{1+i\delta_j}{1-i\delta_j} \right) - \mathrm{Li}_2\left(-\frac{1-i\delta_j}{1+i\delta_j} \right) \right] + \mathcal{O}(\epsilon)
\end{aligned}
\tag{A.5}
$$

We have defined the following:

$$
\begin{aligned}
\Delta_3 &= -(K_1^2)^2 - (K_2^2)^2 - (K_2^2)^2 + 2K_1^2 K_2^2 + 2K_2^2 K_3^2 + 2K_3^2 K_1^2 \\
\delta_1 &= \frac{t_i^{[r]} - t_{i+r}^{[r']} - t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}} \\
\delta_2 &= \frac{-t_i^{[r]} + t_{i+r}^{[r']} - t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}} \\
\delta_3 &= \frac{-t_i^{[r]} - t_{i+r}^{[r']} + t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}}
\end{aligned}
\tag{A.6}
$$

Scalar box integrals:

$$I_4^{4m} = -i(4\pi)^{2-\epsilon} \int \frac{d^4\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{(\ell^2 + i\epsilon)((\ell - K_1)^2 + i\epsilon)((\ell - K_1 - K_2)^2 + i\epsilon)((\ell + K_4)^2 + i\epsilon)}. \tag{A.7}$$

We separate divergent terms from the finite pieces of interest, as explained more fully in section 2. We refer to the finite pieces I_{4F} as finite box integral functions.

$$I_{4;i}^{1m} = \frac{2r_\Gamma}{t_{i-3}^{[2]}t_{i-2}^{[2]}} \frac{1}{\epsilon^2} \left[(-t_{i-3}^{[2]})^{-\epsilon} + (-t_{i-2}^{[2]})^{-\epsilon} - (-t_{i-3}^{[3]})^{-\epsilon} \right] + I_{4F;i}^{1m}.$$

$$I_{4F;i}^{1m} = -\frac{2r_\Gamma}{t_{i-3}^{[2]}t_{i-2}^{[2]}} \left[\mathrm{Li}_2\left(1 - \frac{t_{i-3}^{[3]}}{t_{i-3}^{[2]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i-3}^{[3]}}{t_{i-2}^{[2]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_{i-3}^{[2]}}{t_{i-2}^{[2]}}\right) + \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right].$$

(A.8)

$$I_{4:r;i}^{2m\,e} = \frac{2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+1]} - t_{i-1}^{[r+2]}t_i^{[r]}} \frac{1}{\epsilon^2} \left[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+1]})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i-1}^{[r+2]})^{-\epsilon} \right] + I_{4F:r;i}^{2m\,e}$$

$$I_{4F:r;i}^{2m\,e} = -\frac{2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+1]} - t_{i-1}^{[r+2]}t_i^{[r]}} \left[\mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_i^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i-1}^{[r+2]}}{t_{i-1}^{[r+1]}}\right) \right.$$
$$\left. + \mathrm{Li}_2\left(1 - \frac{t_{i-1}^{[r+2]}}{t_i^{[r+1]}}\right) - \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}t_{i-1}^{[r+2]}}{t_{i-1}^{[r+1]}t_i^{[r+1]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right) + \mathcal{O}(\epsilon) \right].$$

(A.9)

$$I_{4:r;i}^{2m\,h} = \frac{2r_\Gamma}{t_{i-2}^{[2]}t_{i-1}^{[r+1]}} \frac{1}{\epsilon^2} \left[\frac{1}{2}(-t_{i-2}^{[2]})^{-\epsilon} + (-t_{i-1}^{[r+1]})^{-\epsilon} - \frac{1}{2}(-t_i^{[r]})^{-\epsilon} - \frac{1}{2}(-t_{i-2}^{[r+2]})^{-\epsilon} \right] + I_{4F:r;i}^{2m\,h}$$

$$I_{4F:r;i}^{2m\,h} = -\frac{2r_\Gamma}{t_{i-2}^{[2]}t_{i-1}^{[r+1]}} \left[-\frac{1}{2}\ln\left(\frac{t_{i-2}^{[2]}}{t_i^{[r]}}\right)\ln\left(\frac{t_{i-2}^{[2]}}{t_{i-2}^{[r+2]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_{i-2}^{[2]}}{t_{i-1}^{[r+1]}}\right) \right.$$
$$\left. + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i-2}^{[r+2]}}{t_{i-1}^{[r+1]}}\right) + \mathcal{O}(\epsilon) \right].$$

(A.10)

$$I_{4:r:r';i}^{3m} = \frac{2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+r']} - t_i^{[r]}t_{i-1}^{[r+r'+1]}}$$
$$\frac{1}{\epsilon^2} \left[\frac{1}{2}(-t_{i-1}^{[r+1]})^{-\epsilon} + \frac{1}{2}(-t_i^{[r+r']})^{-\epsilon} - \frac{1}{2}(-t_i^{[r]})^{-\epsilon} - \frac{1}{2}(-t_{i-1}^{[r+r'+1]})^{-\epsilon} \right] + I_{4F:r:r';i}^{3m}.$$

$$I_{4F:r:r';i}^{3m} = -\frac{2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+r']} - t_i^{[r]}t_{i-1}^{[r+r'+1]}} \left[-\frac{1}{2}\ln\left(\frac{t_i^{[r+r']}}{t_i^{[r]}}\right)\ln\left(\frac{t_i^{[r+r']}}{t_{i+r}^{[r+r']}}\right) - \frac{1}{2}\ln\left(\frac{t_{i-1}^{[r+1]}}{t_{i+r}^{[r']}}\right)\ln\left(\frac{t_{i-1}^{[r+1]}}{t_{i-1}^{[r+r'+1]}}\right) \right.$$
$$\left. + \frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+r']}}\right) + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i-1}^{[r+r'+1]}}{t_i^{[r+r']}}\right) - \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}t_{i-1}^{[r+r'+1]}}{t_{i-1}^{[r+1]}t_i^{[r+r']}}\right) + \mathcal{O}(\epsilon) \right].$$

(A.11)

The dilogarithm function is defined by $\mathrm{Li}_2(x) = -\int_0^x \ln(1-z)dz/z$.

$$I^{4m} = \frac{1}{a(x_1 - x_2)} \sum_{j=1}^{2} (-1)^j \left(-\frac{1}{2} \ln^2(-x_j) \right)$$

$$- \operatorname{Li}_2 \left(1 + \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon} x_j \right) - \eta \left(-x_k, \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon} \right) \ln \left(1 + \frac{t_{34} - i\epsilon}{t_{13} - i\epsilon} x_j \right) \qquad \text{(A.12)}$$

$$- \operatorname{Li}_2 \left(1 + \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon} x_j \right) - \eta \left(-x_k, \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon} \right) \ln \left(1 + \frac{t_{24} - i\epsilon}{t_{12} - i\epsilon} x_j \right)$$

$$+ \ln(-x_j)(\ln(t_{12} - i\epsilon) + \ln(t_{13} - i\epsilon) - \ln(t_{14} - i\epsilon) - \ln(t_{23} - i\epsilon))) .$$

Here we have defined $t_{ml} \equiv -(K_m + K_{m+1} + \ldots + K_{l-1})^2$.

$$\eta(x, y) = 2\pi i [\vartheta(-\operatorname{Im} x)\vartheta(-\operatorname{Im} y)\vartheta(\operatorname{Im}(xy)) - \vartheta(\operatorname{Im} x)\vartheta(\operatorname{Im} y)\vartheta(-\operatorname{Im}(xy))], \qquad \text{(A.13)}$$

and x_1 and x_2 are the roots of a quadratic polynomial:

$$ax^2 + bx + c + i\epsilon d = a(x - x_1)(x - x_2), \qquad \text{(A.14)}$$

with

$$
\begin{aligned}
a &= t_{24} t_{34}, \\
b &= t_{13} t_{24} + t_{12} t_{34} - t_{14} t_{23}, \\
c &= t_{12} t_{13}, \\
d &= t_{23}.
\end{aligned}
\qquad \text{(A.15)}
$$

Appendix B. Tree Amplitudes with Fermions and Scalars

Here we summarize some results that are useful for our calculations.[7] Take $a = 2$ for a scalar and $a = 1$ for a fermion. (Taking $a = 0$ reproduces the results for all-gluon amplitudes, but these are not needed in this paper.)

$$A(4_{F/S}^+, 5^+, 6^+, 1^-, 2^-, 3_{F/S}^-) = I_{12|3456} + I_{1234|56}$$

$$= \frac{\langle 3|1 + 2|6]^3}{[6\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle P_{345}^2 \langle 5|6 + 1|2]} \left(-\frac{\langle 4|1 + 2|6]}{\langle 3|1 + 2|6]} \right)^a \qquad \text{(B.1)}$$

$$+ \frac{\langle 1|5 + 6|4]^3}{[2\,3][3\,4]\langle 5\,6\rangle\langle 6\,1\rangle P_{561}^2 \langle 5|6 + 1|2]} \left(\frac{\langle 1|5 + 6|3]}{\langle 1|5 + 6|4]} \right)^a$$

---

[7] Some of these results have appeared in [28,29].

Similarly we have

$$A(4^-_{F/S}, 5^+, 6^+, 1^-, 2^-, 3^+_{F/S}) = I_{12|3456} + +I_{1234|56}$$

$$= \frac{\langle 4|1+2|6]^4}{[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle P^2_{345}\langle 5|6+1|2]\langle 3|1+2|6]} \left(\frac{\langle 3|1+2|6]}{\langle 4|1+2|6]} \right)^a$$

$$+ \frac{\langle 1|5+6|3]^4}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P^2_{561}\langle 5|6+1|2]\langle 1|5+6|4]} \left(-\frac{\langle 1|5+6|4]}{\langle 1|5+6|3]} \right)^a \quad \text{(B.2)}$$

Notice that for the split $I_{123|456}$ with a fermionic line between the two vertices, the amplitude is zero (unlike the internal gluon case).

$$A(1^+_{F/S}, 2^-, 3^+, 4^+, 5^-, 6^-_{F/S}) = I_{12|3456} + I_{612|345} + I_{5612|34}$$

$$= \frac{[1\ 3]^4\langle 5\ 6\rangle^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P^2_{123}\langle 4|P_{123}|1]\langle 6|P_{123}|3]} \left(-\frac{\langle 5|P_{123}|3]}{[1\ 3]\langle 5\ 6\rangle} \right)^a$$

$$+ \frac{\langle 6\ 2\rangle^4[3\ 4]^4}{\langle 6\ 1\rangle\langle 1\ 2\rangle[3\ 4][4\ 5]\langle 2|P_{612}|5]\langle 6|P_{612}|3]P^2_{612}} \left(-\frac{\langle 2\ 1\rangle}{\langle 2\ 6\rangle} \right)^a \quad \text{(B.3)}$$

$$+ \frac{\langle 2|P_{234}|1]^4}{\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]P^2_{234}\langle 4|P_{234}|1]\langle 2|P_{234}|5]} \left(-\frac{\langle 2|P_{234}|6]}{\langle 2|P_{234}|1]} \right)^a$$

$$A(1^-_{F/S}, 2^-, 3^+, 4^+, 5^-, 6^+_{F/S}) = I_{23|4561} + I_{123|456} + I_{6123|45}$$

$$= \frac{[3\ 4]^4\langle 5\ 1\rangle^4}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P^2_{234}\langle 1|P_{234}|4]\langle 5|P_{234}|2]} \left(\frac{\langle 5\ 6\rangle}{\langle 5\ 1\rangle} \right)^a$$

$$+ \frac{\langle 1\ 2\rangle^4[4\ 6]^4}{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\langle 3|P_{456}|6]\langle 1|P_{456}|4]P^2_{456}} \left(\frac{\langle 2|P_{456}|4]}{\langle 2\ 1\rangle[6\ 4]} \right)^a \quad \text{(B.4)}$$

$$+ \frac{\langle 5|P_{345}|6]^4}{\langle 3\ 4\rangle\langle 4\ 5\rangle[6\ 1][1\ 2]P^2_{345}\langle 5|P_{345}|2]\langle 3|P_{345}|6]} \left(-\frac{\langle 5|P_{345}|1]}{\langle 5|P_{345}|6]} \right)^a$$

$$A(1^-_{F/S}, 2^-, 3^+, 4^-, 5^+, 6^+_{F/S}) = I_{12|3456} + I_{612|345} + I_{5612|34}$$

$$= \frac{\langle 4|P_{123}|3]^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P^2_{123}\langle 4|P_{123}|1]\langle 6|P_{123}|3]} \left(\frac{[1\ 3]\langle 4\ 6\rangle}{\langle 4|P_{123}|3]} \right)^a$$

$$+ \frac{\langle 1\ 2\rangle^4[3\ 5]^4}{\langle 6\ 1\rangle\langle 1\ 2\rangle[3\ 4][4\ 5]\langle 2|P_{612}|5]\langle 6|P_{612}|3]P^2_{612}} \left(\frac{\langle 2\ 6\rangle}{\langle 2\ 1\rangle} \right)^a \quad \text{(B.5)}$$

$$+ \frac{[5\ 6]^4\langle 4\ 2\rangle^4}{\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]P^2_{234}\langle 4|P_{234}|1]\langle 2|P_{234}|5]} \left(-\frac{[5\ 1]}{[5\ 6]} \right)^a$$

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$$A(1^+_{F/S}, 2^-, 3^+, 4^-, 5^+, 6^-_{F/S}) = I_{12|3456} + I_{612|345} + I_{5612|34}$$

$$= \frac{[1\ 3]^4\langle 4\ 6\rangle^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle P_{123}^2\langle 4|P_{123}|1]\langle 6|P_{123}|3]}\left(-\frac{\langle 4|P_{123}|3]}{[1\ 3]\langle 4\ 6\rangle}\right)^a$$

$$+ \frac{\langle 6\ 2\rangle^4[3\ 5]^4}{\langle 6\ 1\rangle\langle 1\ 2\rangle[3\ 4][4\ 5]\langle 2|P_{612}|5]\langle 6|P_{612}|3]P_{612}^2}\left(-\frac{\langle 2\ 1\rangle}{\langle 2\ 6\rangle}\right)^a \qquad (B.6)$$

$$+ \frac{[5\ 1]^4\langle 4\ 2\rangle^4}{\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]P_{234}^2\langle 4|P_{234}|1]\langle 2|P_{234}|5]}\left(\frac{[5\ 6]}{[5\ 1]}\right)^a$$

The NMHV tree-level amplitude with adjacent negative helicities is given by

$$A(4^+_{F/S}, 5^+, ..., n^+, 1^-, 2^-, 3^-_{F/S})$$

$$= -\frac{1}{\prod_{i=3}^n\langle i\ i+1\rangle}\sum_{j=0}^{n-5}\frac{\langle n-j-1\ \ n-j\rangle\langle 1|K_2^{[n-j-2]}K_3^{[n-j-3]}|3\rangle^3}{t_2^{[n-j-2]}t_3^{[n-j-3]}\langle n-j|K_2^{[n-j-2]}|2]\langle n-j-1|K_2^{[n-j-2]}|2]} \qquad (B.7)$$

$$\times \left(-\frac{\langle 1|K_2^{[n-j-2]}K_3^{[n-j-3]}|4\rangle}{\langle 1|K_2^{[n-j-2]}K_3^{[n-j-3]}|3\rangle}\right)^a$$

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